JEE Main 2023 24th Jan Shift 2 Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Section - A (Physics)

1. The electric potential at the centre of two concentric half rings of radii R_1 and R_2, having the same linear charge density λ is:

(1) $\frac{2\lambda}{\epsilon_0}$
(2) $\frac{\lambda}{2\epsilon_0}$
(3) $\frac{\lambda}{4\epsilon_0}$
(4) $\frac{\lambda}{\epsilon_0}$

Correct Answer: (2) $\frac{\lambda}{2\epsilon_0}$

Solution: The potential at the centre of two concentric rings is the sum of the potentials due to each ring. The formula for the potential at the centre is:

$$V = \frac{\lambda \pi R_1}{4\pi\epsilon_0 R_1} + \frac{\lambda \pi R_2}{4\pi\epsilon_0 R_2}$$

Simplifying this, we get the result:

$$V = \frac{\lambda}{2\epsilon_0}$$

Quick Tip

For concentric rings, calculate the individual potentials due to each ring and then sum them. The potential at the centre is the result of contributions from both rings.

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2. Let γ_1 be the ratio of molar specific heat at constant pressure and molar specific heat at constant volume of a monoatomic gas and γ_2 be the similar ratio of diatomic gas. Considering the diatomic gas molecule as a rigid rotator, the ratio $\frac{\gamma_1}{\gamma_2}$ is:

(1) $\frac{27}{35}$

(2) $\frac{35}{27}$

(3) $\frac{21}{25}$

(4) $\frac{25}{21}$

Correct Answer: (3) $\frac{21}{25}$

Solution: For a monoatomic gas, $\gamma_1 = \frac{5}{3}$. For a diatomic gas at low temperatures, $\gamma_2 = \frac{7}{5}$. Thus,

$$\frac{\gamma_1}{\gamma_2} = \frac{\frac{5}{3}}{\frac{7}{5}} = \frac{25}{21}$$

Quick Tip

For gases, use the formula for γ based on the degrees of freedom. For monoatomic and diatomic gases, the values of γ differ significantly.

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3. An α-particle, a proton and an electron have the same kinetic energy. Which one of the following is correct in case of their De-Broglie wavelength:

(1) $\lambda_\alpha > \lambda_p > \lambda_e$

(2) $\lambda_\alpha < \lambda_p < \lambda_e$

(3) $\lambda_\alpha = \lambda_p = \lambda_e$

(4) $\lambda_\alpha > \lambda_p < \lambda_e$

Correct Answer: (2) $\lambda_\alpha < \lambda_p < \lambda_e$

Solution: De-Broglie wavelength is given by:

$$\lambda = \frac{h}{\sqrt{2mK}}$$

Since K (kinetic energy) is the same for all three particles, the wavelength depends on their masses. The mass of the α-particle is the largest, followed by the proton, and then the electron. Thus:

$$\lambda_\alpha < \lambda_p < \lambda_e$$

Quick Tip

De-Broglie wavelength is inversely proportional to the square root of the mass. The lighter the particle, the larger its wavelength.

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4. If the distance of the earth from Sun is 1.5×10^6 km, then the distance of an imaginary planet from Sun, if its period of revolution is 2.83 years, is:

(1) 6×10^7 km

(2) 6×10^6 km

(3) 3×10^6 km

(4) 3×10^7 km

Correct Answer: (3) 3×10^6 km

Solution: We use Kepler's third law:

$$T^2 \propto R^3$$

Let $T_1 = 1$ year and $R_1 = 1.5 \times 10^6$ km. The period for the imaginary planet is given as $T_2 = 2.83$ years, and we need to find R_2. Using the relation:

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3$$

Substitute the values to find $R_2 = 3 \times 10^6$ km.

Quick Tip

Kepler's third law relates the orbital period of a planet to its distance from the Sun. Use this law to solve problems involving planetary motion.

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5. Match List I with List II:

LIST I	LIST II
A. AM Broadcast	I. 88-108 MHz
B. FM Broadcast	II. 540-1600 KHz
C. Television	III. 3.7-4.2 GHz
D. Satellite Communication	IV. 54 MHz, 590 MHz

Choose the correct answer from the options given below:

(1) A-II, B-I, C-IV, D-III

(2) A-IV, B-III, C-I, D-II

(3) A-II, B-III, C-I, D-IV

(4) A-I, B-II, C-IV, D-III

Correct Answer: (3) A-II, B-III, C-I, D-IV

Solution:

- AM Broadcast corresponds to $540 - 1600$ KHz, so A matches with II.
- FM Broadcast corresponds to $88 - 108$ MHz, so B matches with I.
- Television corresponds to $54 - 890$ MHz, so C matches with IV.
- Satellite Communication corresponds to $3.7 - 4.2$ GHz, so D matches with III.

Thus, the correct matching is A-II, B-III, C-I, D-IV.

Quick Tip

Matching questions test your knowledge of the frequency ranges of different communication systems. Review standard frequency bands for AM, FM, TV, and satellite communication.

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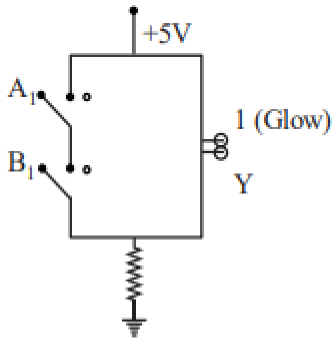
6. The logic gate equivalent to the given circuit diagram is:



(1) OR

(2) NAND

(3) NOR

(4) AND

Correct Answer: (2) NAND

Solution: The given circuit has two inputs A_1 and B_1, and its output is denoted by Y. From the truth table, we observe that the output corresponds to a NAND gate. The truth table is:

A_1	B_1	Y
0	0	1
0	1	1
1	0	1
1	1	0

The logic gate that produces this output is the NAND gate, and the equation for the output is:

$$Y = A_1 \text{ NAND } B_1$$

Quick Tip

To identify logic gates, use the truth table. For a NAND gate, the output is 1 for all combinations of inputs except when both inputs are 1.

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7. A long solenoid is formed by winding 70 turns cm^{-1}. If 2.0 A current flows, then the magnetic field produced inside the solenoid is:

(1) $1232 \times 10^{-4} T$

(2) $176 \times 10^{-4}\,T$

(3) $352 \times 10^{-4}\,T$

(4) $88 \times 10^{-4}\,T$

Correct Answer: (2) $176 \times 10^{-4}\,T$

Solution: The magnetic field produced inside a solenoid is given by the formula:

$$B = \mu_0 n I$$

where n is the number of turns per unit length, I is the current, and $\mu_0 = 4\pi \times 10^{-7}\,T \cdot m/A$ is the permeability of free space. Given: $n = 70\,\text{turns/cm} = 70 \times 10^2\,\text{turns/m}$ $I = 2.0\,A$

Substitute the values into the formula:

$$B = 4\pi \times 10^{-7} \times 70 \times 10^2 \times 2 = 176 \times 10^{-4}\,T$$

Quick Tip

The magnetic field inside a solenoid is directly proportional to the current and the number of turns per unit length. Use this formula to quickly calculate the magnetic field.

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8. Given below are two statements:

Statement I: Acceleration due to earth's gravity decreases as you go 'up' or 'down' from earth's surface.

Statement II: Acceleration due to earth's gravity is the same at a height h and depth d from earth's surface, if $h = d$.

In the light of above statements, choose the most appropriate answer from the options given below:

(1) Statement I is incorrect but Statement II is correct

(2) Both Statement I and Statement II are incorrect

(3) Statement I is correct but Statement II is incorrect

(4) Both Statement I and Statement II are correct

Correct Answer: (3) Statement I is correct but Statement II is incorrect

Solution: Statement I: This statement is correct. As you move away from the surface or go deeper into the earth, the acceleration due to gravity decreases.

Statement II: This statement is incorrect. The acceleration due to gravity is the same at a height h and depth d from the earth's surface only if the height h is equal to the depth d, which is not generally true.

The formula for acceleration due to gravity at a height or depth is:

$$g' = g\left(\frac{1 + h/R}{1 + d/R}\right)$$

where g is the acceleration due to gravity at the surface, R is the radius of the earth, and h and d are the height and depth, respectively.

Thus, the correct answer is that Statement I is correct, but Statement II is incorrect.

Quick Tip

Gravity decreases with height and depth, but the exact relationship involves both the height above and the depth below the surface.

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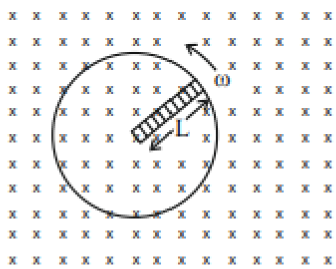
9. A metallic rod of length L is rotated with an angular speed of ω normal to a uniform magnetic field B about an axis passing through one end of the rod as shown in the figure. The induced emf will be:



(1) $\frac{1}{4}B^2L\omega$

(2) $\frac{1}{4}BL\omega$

(3) $\frac{1}{2}BL^2\omega$

(4) $\frac{1}{2}B^2L^2\omega$

Correct Answer: (3) $\frac{1}{2}BL^2\omega$

Solution: The induced emf in a rotating rod in a magnetic field can be determined using the following relationship for the induced emf:

$$\text{emf} = \int_0^L B\,v\,dx$$

where $v = \omega x$ is the linear velocity at a distance x from the axis of rotation, and L is the length of the rod.

Substitute $v = \omega x$ into the equation:

$$\text{emf} = \int_0^L B\,(\omega x)\,dx = B\omega \int_0^L x\,dx$$

$$\text{emf} = B\omega \left[\frac{x^2}{2}\right]_0^L = \frac{1}{2}BL^2\omega$$

Thus, the induced emf is $\frac{1}{2}BL^2\omega$.

Quick Tip

To calculate the induced emf in a rotating rod, integrate the velocity over the length of the rod. The linear velocity depends on the distance from the axis of rotation.

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10. When a beam of white light is allowed to pass through a convex lens parallel to the principal axis, the different colours of light converge at different points on the principal axis after refraction. This is called:

(1) Scattering

(2) Chromatic aberration

(3) Spherical aberration

(4) Polarisation

Correct Answer: (2) Chromatic aberration

Solution: When a beam of white light passes through a convex lens, different wavelengths (colours) of light refract by different amounts due to their varying refractive indices. This phenomenon, where different colours converge at different points along the principal axis, is known as chromatic aberration.

Quick Tip

Chromatic aberration occurs because the lens refracts different colours of light at different angles. This can be minimized using achromatic lenses.

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11. The frequency (v) of an oscillating liquid drop may depend upon radius (r) of the drop, density (ρ) of liquid and the surface tension (s) of the liquid as:

$$v = r^a \rho^b s^c$$

The values of a, b, and c respectively are:

(1) $\left(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

(2) $\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

(3) $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$

(4) $\left(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

Correct Answer: (1) $\left(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

Solution: We start with the dimensional analysis. The frequency of oscillation is given as:

$$[v] = [L^0 T^{-1}]$$

Now, the units of r, ρ, and s are:

$$[r] = L, \quad [\rho] = ML^{-3}, \quad [s] = MT^{-2}$$

Using the dimensional formula $T^1 = M^a L^b T^c$, we get the following system of equations for the exponents:

$$c = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad a = \frac{3}{2}$$

Thus, $a = \frac{3}{2}, b = -\frac{1}{2}, c = \frac{1}{2}$.

Quick Tip

Dimensional analysis helps in determining the relation between physical quantities. Ensure the dimensions on both sides of the equation are balanced for accurate results.

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12. A body of mass 200g is tied to a spring of spring constant 12.5 N/m, while the other end of the spring is fixed at point O. If the body moves about O in a circular path on a smooth horizontal surface with constant angular speed 5 rad/s, then the ratio of extension in the spring to its natural length will be:

(1) 1:2

(2) 1:1

(3) 2:3

(4) 2:5

Correct Answer: (3) 2:3

Solution: Let the natural length of the spring be L_0 and the extension be x. For the circular motion of the body, the centripetal force is provided by the restoring force of the spring:

$$F = m\omega^2 r = kx$$

where $\omega = 5\,\text{rad/s}$, $m = 200\,\text{g} = 0.2\,\text{kg}$, and $k = 12.5\,\text{N/m}$. The total length of the spring is $L_0 + x$, and the centripetal force is given by:

$$F = 0.2 \times (5)^2 = 12.5x$$

Simplifying this, we get:

$$12.5 = 12.5x \quad \Rightarrow \quad x = \frac{2}{3}L_0$$

Thus, the ratio of the extension to the natural length is $\frac{2}{3}$, or 2:3.

Quick Tip

For problems involving circular motion, the restoring force from the spring provides the necessary centripetal force. Use this relation to calculate the extension of the spring.

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13. A cell of emf 90 V is connected across a series combination of two resistors each of $100\,\Omega$ resistance. A voltmeter of resistance 400 Ω is used to measure the potential difference across each resistor. The reading of the voltmeter will be:

(1) 40 V

(2) 45 V

(3) 80 V

(4) 90 V

Correct Answer: (1) 40 V

Solution: The resistors are connected in series, so the equivalent resistance R_{eq} is:

$$R_{\text{eq}} = 100\,\Omega + 100\,\Omega = 200\,\Omega$$

The total current i in the circuit is:

$$i = \frac{90\,\text{V}}{200\,\Omega} = \frac{1}{2}\,\text{A}$$

Now, the voltage across each resistor is given by:

$$V = i \times R = \frac{1}{2} \times 100 = 50\,\text{V}$$

However, the voltmeter has a resistance of 400 Ω, and it is connected across one of the resistors. The effective resistance of the combination is:

$$R_{\text{eq}} = \frac{100 \times 400}{100 + 400} = 80\,\Omega$$

So, the total current in the circuit becomes:

$$i = \frac{90\,\text{V}}{180\,\Omega} = \frac{1}{2}\,\text{A}$$

Finally, the reading of the voltmeter is:

$$V_{\text{reading}} = \frac{1}{2} \times 400 \times \frac{100}{500} = 40\,\text{V}$$

Quick Tip

When measuring potential difference across a resistor with a voltmeter, account for the internal resistance of the voltmeter, which forms a parallel combination with the resistor.

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14. The electric field and magnetic field components of an electromagnetic wave going through vacuum is described by:

$$E_x = E_0 \sin(kz - \omega t)$$

$$B_y = B_0 \sin(kz - \omega t)$$

Then the correct relation between E_0 and B_0 is given by:

(1) $kE_0 = B_0$

(2) $E_0 B_0 = ck$

(3) $\omega E_0 = kB_0$

(4) $E_0 = kB_0$

Correct Answer: (1) $kE_0 = B_0$

Solution: For an electromagnetic wave in a vacuum, the electric and magnetic fields are related by the following equation:

$$E_0 = cB_0$$

where c is the speed of light. Additionally, the wave number k and the angular frequency ω are related to the speed of light as:

$$c = \frac{\omega}{k}$$

Thus, combining these relations, we obtain:

$$kE_0 = B_0$$

Quick Tip

For electromagnetic waves, the electric and magnetic fields are perpendicular to each other and the direction of wave propagation. Use the relationship $E_0 = cB_0$ to find the relationship between field components.

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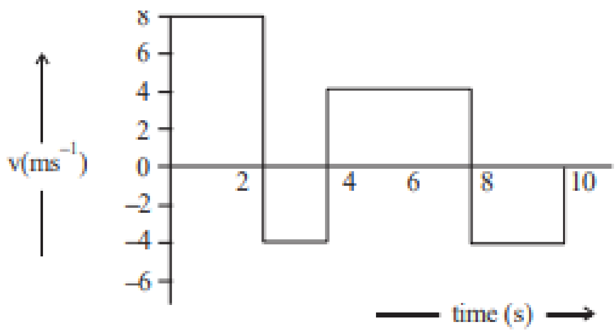
15. The velocity-time graph of a body moving in a straight line is shown in the figure.



The ratio of displacement to distance travelled by the body in time 0 to 10s is:

(1) 1:1

(2) 1:4

(3) 1:2

(4) 1:3

Correct Answer: (4) 1:3

Solution:

The displacement is the area under the velocity-time graph.

The displacement is calculated as:

$$\text{Displacement} = \text{Area} = 16 - 8 + 16 - 8 = 16\,\text{m}$$

The total distance travelled is the sum of the areas, which is:

$$\text{Distance} = \sum(\text{area}) = 48\,\text{m}$$

Thus, the ratio of displacement to distance is:

$$\frac{\text{Displacement}}{\text{Distance}} = \frac{16}{48} = \frac{1}{3}$$

Quick Tip

When dealing with velocity-time graphs, the area under the graph represents displacement, and the total distance is the sum of the absolute areas.

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16. Given below are two statements: one is labelled as Assertion A and the other is labelled as Reason R

Assertion A: Steel is used in the construction of buildings and bridges.

Reason R: Steel is more elastic and its elastic limit is high.

In the light of above statements, choose the most appropriate answer from the options given below:

(1) Both A and R are correct but R is NOT the correct explanation of A

(2) A is not correct but R is correct

(3) Both A and R are correct and R is the correct explanation of A

(4) A is correct but R is not correct

Correct Answer: (3) Both A and R are correct and R is the correct explanation of A

Solution: Assertion A is correct because steel is indeed used in the construction of buildings and bridges due to its strength. Reason R is also correct because steel is more elastic and has a high elastic limit, which makes it ideal for construction purposes where high strength and flexibility are needed.

Since Reason R provides the correct explanation for Assertion A, the correct answer is option (3).

Quick Tip

In assertion-reasoning questions, always check whether the reason correctly explains the assertion.

17. Given below are two statements: one is labelled as Assertion A and the other is labelled as Reason R

Assertion A: A pendulum clock when taken to Mount Everest becomes fast.

Reason R: The value of g (acceleration due to gravity) is less at Mount Everest than its value on the surface of the Earth.

In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Both A and R are correct but R is NOT the correct explanation of A

(2) Both A and R are correct and R is the correct explanation of A

(3) A is not correct but R is correct

(4) A is correct but R is not correct

Correct Answer: (3) A is not correct but R is correct

Solution: Assertion A is incorrect because a pendulum clock becomes slower at higher altitudes due to the lower value of g. Reason R is correct as the value of gravity decreases with altitude, which is why the pendulum clock runs slower on Mount Everest. Thus, A is incorrect, but R is correct.

Quick Tip

In such problems, always check the correctness of both the assertion and the reason before selecting the appropriate answer.

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18. A photon is emitted in transition from $n = 4$ to $n = 1$ level in hydrogen atom. The corresponding wavelength for this transition is (given, $h = 4 \times 10^{-15}$ eV $\cdot$ s):

(1) 94.1 nm

(2) 941 nm

(3) 97.4 nm

(4) 99.3 nm

Correct Answer: (1) 94.1 nm

Solution: The energy of the photon is given by the difference in energy levels:

$$E = h\nu = hc\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Substitute the values $n_1 = 1$, $n_2 = 4$, and $h = 4 \times 10^{-15}\,\text{eV} \cdot \text{s}$, $c = 3 \times 10^8$ m/s:

$$E = 4 \times 10^{-15} \times 3 \times 10^8 \times \left(\frac{1}{1^2} - \frac{1}{4^2}\right) = 13.6\,\text{eV}$$

The corresponding wavelength λ is given by:

$$\lambda = \frac{hc}{E} = \frac{4 \times 10^{-15} \times 3 \times 10^8}{13.6} = 94.1\,\text{nm}$$

Quick Tip

For transitions between energy levels in hydrogen, use the Rydberg formula to find the energy difference and subsequently the wavelength.

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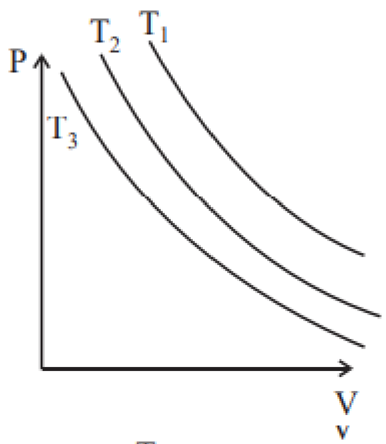
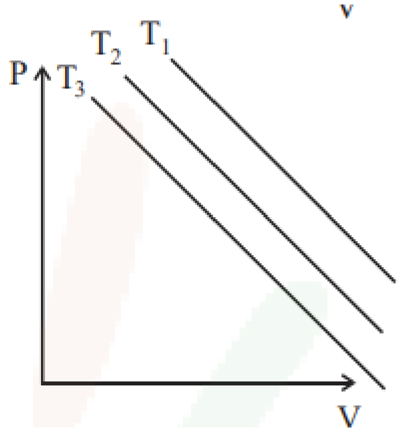
19. In an Isothermal change, the change in pressure and volume of a gas can be represented for three different temperatures; $T_3 > T_2 > T_1$ as:



(1)



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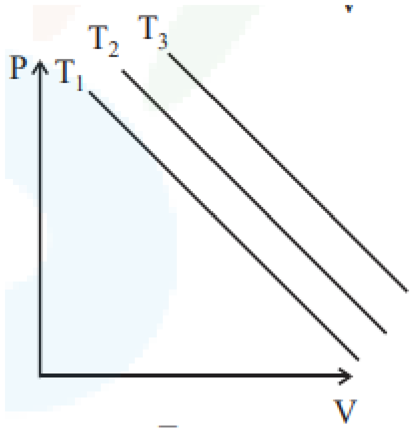

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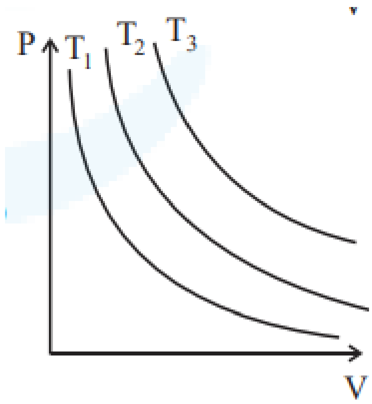
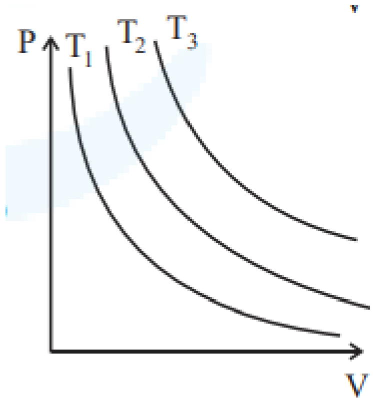


(4)



Correct Answer: (4)

Solution: For an isothermal process, the $P - V$ graph is represented as a rectangular hyperbola, with the pressure inversely proportional to volume. The higher the temperature, the greater the volume for the same pressure, which is why $T_3 > T_2 > T_1$ leads to the following graph. The dotted line in the graph represents an isobaric line.

Quick Tip

In an isothermal process, temperature remains constant, so the product of pressure and volume is constant. This relationship is represented as a hyperbolic curve in the $P - V$ graph.

20. If two vectors P $= \hat{i} + 2\hat{j} + m\hat{k}$ and Q $= 4\hat{i} - 2\hat{j} + \hat{k}$ are perpendicular to each other, then the value of m will be:

(1) 1

(2) -1

(3) 3

(4) 2

Correct Answer: (4) 2

Solution: For two vectors to be perpendicular, their dot product must be zero. So,

$$\mathbf{P} \cdot \mathbf{Q} = 0$$

$$(i + 2j + mk) \cdot (4i - 2j + k) = 0$$

$$4 - 4 + m = 0$$

$$m = 2$$

Quick Tip

To check if two vectors are perpendicular, calculate their dot product. If it equals zero, the vectors are perpendicular.

Section - B (Physics)

21. A uniform solid cylinder with radius R and length L has moment of inertia I_1, about the axis of the cylinder. A concentric solid cylinder of radius $R' = \frac{R}{2}$ and length $L' = \frac{L}{2}$ is carved out of the original cylinder. If I_2 is the moment of inertia of the carved-out portion of the cylinder, then $\frac{I_1}{I_2}$ is:

Solution: The moment of inertia of a solid cylinder about its axis is given by:

$$I = \frac{1}{2}mR^2$$

where m is the mass and R is the radius. For the full cylinder:

$$I_1 = \frac{1}{2}mR^2$$

For the carved-out cylinder, the radius is $R' = \frac{R}{2}$ and the length is $L' = \frac{L}{2}$, so:

$$I_2 = \frac{1}{2}m'R'^2 = \frac{1}{2}\left(\frac{m}{4}\right)\left(\frac{R}{2}\right)^2 = \frac{1}{2} \times \frac{m}{4} \times \frac{R^2}{4} = \frac{mR^2}{32}$$

Thus,

$$\frac{I_1}{I_2} = \frac{\frac{1}{2}mR^2}{\frac{mR^2}{32}} = 32$$

Quick Tip

For concentric cylinders, calculate the moment of inertia using the formula $I = \frac{1}{2}mR^2$, and apply the ratio of masses and radii to determine the moment of inertia of the carved-out portion.

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22. A mass m attached to the free end of a spring executes SHM with a period of 1s. If the mass is increased by 3 kg, the period of oscillation increases by one second, the value of mass m is:

Solution: The formula for the period T of a mass-spring system is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where m is the mass and k is the spring constant. Given that the period increases by 1 second when the mass increases by 3 kg, we can set up the equation for the new and original periods:

$$T' = T + 1 = 2\pi\sqrt{\frac{m+3}{k}}$$

Substituting for the initial period $T = 1$ second:

$$1 + 1 = 2\pi\sqrt{\frac{m+3}{k}} \quad \Rightarrow \quad m = 1\,\text{kg}$$

Quick Tip

For SHM, the period is proportional to the square root of the mass. Use this relationship to set up equations for the initial and final periods and solve for the unknown mass.

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23. The energy released per fission of nucleus of ^{240}X is 200 MeV. The energy released if all the atoms in 120g of pure ^{240}X undergo fission is:

Solution: Given that the energy released per fission of a nucleus is 200 MeV, and the number of moles is:

$$\text{No. of moles} = \frac{120}{240} = \frac{1}{2}\,\text{moles}$$

The number of molecules is:

$$\text{No. of molecules} = \frac{1}{2}N_A$$

where $N_A = 6 \times 10^{23}$ is Avogadro's number. The total energy released is:

$$E = \frac{1}{2} \times 6 \times 10^{23} \times 200 = 6 \times 10^{25}\,\text{MeV}$$

Quick Tip

Use Avogadro's number to find the total number of molecules, and then multiply by the energy released per fission event to find the total energy released.

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24. A parallel plate capacitor with air between the plates has a capacitance of 15 pF. The separation between the plates becomes twice and the space between them is filled with a material of dielectric constant 3.5. Then the capacitance becomes:

Solution: The capacitance of a parallel plate capacitor is given by:

$$C = \frac{\varepsilon_0 A}{d}$$

where A is the area of the plates, ε_0 is the permittivity of free space, and d is the distance between the plates. When the dielectric constant is added, the capacitance becomes:

$$C' = \kappa C$$

Given that $\kappa = 3.5$, and the separation between the plates becomes twice, the capacitance becomes:

$$C' = \frac{15}{2} \times 3.5 = 4\,\text{pF}$$

Quick Tip

When the distance between plates of a capacitor is doubled, the capacitance is halved. Additionally, the dielectric constant increases the capacitance.

25. A body of mass 1 kg begins to move under the action of a time dependent force $\mathbf{F} = \hat{i} + 3t\hat{j}$ N. where $\hat{i}$ and $\hat{j}$ are the unit vectors along x and y axis. The power developed by above force, at the time $t = 2$ s will be:

Solution: The power developed is given by the dot product of force and velocity:

$$P = \mathbf{F} \cdot \mathbf{v}$$

We are given that the force $\mathbf{F} = \hat{i} + 3t\hat{j}$, so the acceleration is:

$$\frac{d\mathbf{v}}{dt} = \hat{i} + 3t\hat{j}$$

Integrating with respect to time:

$$\mathbf{v} = \int \left(\hat{i} + 3t\hat{j}\right) dt = t\hat{i} + \frac{3t^2}{2}\hat{j}$$

At $t = 2$ s, the velocity is:

$$\mathbf{v} = 2\hat{i} + 6\hat{j}$$

The power is:

$$P = \mathbf{F} \cdot \mathbf{v} = (1\hat{i} + 6\hat{j}) \cdot (2\hat{i} + 6\hat{j}) = 2 + 6 \times 6 = 100\,\text{W}$$

Quick Tip

When calculating power, use the dot product of the force and velocity vectors. The units of power are watts (W), which are joules per second (J/s).

---

26. If a copper wire is stretched to increase its length by 20%. The percentage increase in resistance of the wire is:

Solution: The resistance R of a wire is proportional to its length L, so:

$$R \propto L$$

Thus, if the length increases by 20

$$\Delta R = 24 + 20 = 44\% \text{ increase in resistance.}$$

Quick Tip

Resistance of a wire is directly proportional to its length and inversely proportional to its cross-sectional area. If the wire is stretched, the resistance increases.

---

27. A single turn current loop in the shape of a right angle triangle with sides 5 cm, 12 cm, 13 cm is carrying a current of 2A. The loop is in a uniform magnetic field of magnitude 0.75 T whose direction is parallel to the current in the 13 cm side of the loop. The magnitude of the magnetic force on the 5 cm side will be:

Solution: The magnetic force on a current-carrying conductor is given by:

$$F = ILB\sin\theta$$

where $I = 2\,\text{A}$, $L = 5\,\text{cm} = 0.05\,\text{m}$, $B = 0.75\,\text{T}$, and $\theta = 90°$. Thus, the force is:

$$F = (2)(0.05)(0.75) = 9\,\text{N}$$

Quick Tip

For calculating the magnetic force on a current-carrying wire, ensure you know the current, length of the wire in the field, magnetic field strength, and the angle between the wire and field.

---

28. Three identical resistors with resistance $R = 12\,\Omega$ and two identical inductors with self inductance $L = 5\,\text{mH}$ are connected to an ideal battery with emf of 12 V as shown in figure. The current through the battery long after the switch has been closed will be:

Solution: After a long time, the inductor behaves as a resistance-less path, so the current through the resistors is governed by the total resistance. The equivalent resistance is:

$$R_{\text{eq}} = \frac{R}{3} = 4\,\Omega$$

Thus, the current through the battery is:

$$I = \frac{12}{R_{\text{eq}}} = \frac{12}{4} = 3\,\text{A}$$

Quick Tip

When inductors are connected in a DC circuit long after the switch is closed, they act like short circuits because the current through an inductor stabilizes.

---

29. A convex lens of refractive index 1.5 and focal length 18 cm in air is immersed in water. The change in focal length of the lens will be: (Given refractive index of water $= \frac{4}{3}$)

Solution: The formula for the focal length of a lens is:

$$\frac{1}{f_{\text{air}}} = \left(\frac{\mu_l}{\mu_{\text{air}}} - 1\right)\left(\frac{2}{R}\right)$$

where μ_l is the refractive index of the lens and $\mu_{\text{air}} = 1$. For the lens immersed in water, the refractive index of water is $\frac{4}{3}$, so:

$$\frac{1}{f_{\text{water}}} = \left(\frac{\mu_l}{\mu_{\text{water}}} - 1\right)\left(\frac{2}{R}\right)$$

Thus,

$$f_{\text{water}} = 72\,\text{cm}$$

The change in focal length is:

$$72 - 54 = 18\,\text{cm}$$

Quick Tip

When a lens is immersed in a medium, the refractive index of the medium affects the focal length. In water, the focal length increases as the refractive index of water is less than that of air.

---

30. A spherical ball of radius 1mm and density 10.5 g/cc is dropped in glycerine of coefficient of viscosity 9.8 poise and density 1.5 g/cc. Viscous force on the ball when it attains constant velocity is 3696 $\times 10^{-7}$ N. The value of x is:

Solution: When the ball attains terminal velocity, the force balance is given by:

$$F = mg - F_b$$

where F_b is the buoyant force. The expression for the viscous force is:

$$F = V\rho_g g$$

The volume of the ball is $V = \frac{4}{3}\pi r^3$. Substituting the values:

$$F = \left(\frac{4}{3}\pi(10^{-3})^3 \times 9.8 \times (10.5 - 1.5) \times 10^3\right) = 3696 \times 10^{-7}\,\text{N}$$

Thus, $x = 7$.

Quick Tip

For objects moving through a fluid, use the viscous force equation for terminal velocity. Remember to use the volume of the sphere to calculate the force and density values.

Section - A (Chemistry)

31. Which one amongst the following are good oxidizing agents? A. Sm^{2+} B. Ce^{2+} C. Ce^{4+} D. Tb^{4+}

Choose the most appropriate answer from the options given below:

(1) C only

(2) D only

(3) A and B only

(4) C and D only

Correct Answer: (4) C and D only

Solution: The ions Ce^{4+} and Tb^{4+} are strong oxidizing agents, as they have a high tendency to gain electrons and reduce to their respective lower oxidation states.

Quick Tip

Oxidizing agents are substances that gain electrons and in the process, get reduced. Look for elements with high positive oxidation states.

---

32. What is the number of unpaired electrons(s) in the highest occupied molecular orbital of the following species: $N_2, N_2^+, O_2, O_2^?$

(1) 0, 1, 2, 1

(2) 2, 1, 2, 1

(3) 1, 0, 1, 0

(4) 2, 1, 0, 1

Correct Answer: (1) 0, 1, 2, 1

Solution:

- N_2: σ^*1s and $\sigma 2p^2$ are fully paired, so no unpaired electrons.

- N_2^+: Loss of one electron leaves one unpaired electron in the π^*2p orbital.

- O_2: The π^*2p orbitals have two unpaired electrons.

- O_2^+: Removal of one electron results in one unpaired electron.

Quick Tip

Use molecular orbital theory to determine the number of unpaired electrons by observing the electron configuration of each species.

---

33. Which of the following cannot be explained by crystal field theory?

(1) The order of spectrochemical series

(2) Magnetic properties of transition metal complexes

(3) Colour of metal complexes

(4) Stability of metal complexes

(1) The order of spectrochemical series

(2) Magnetic properties of transition metal complexes

(3) Colour of metal complexes

(4) Stability of metal complexes

Correct Answer: (1)

Solution: Crystal field theory (CFT) explains the magnetic properties, colour, and stability of metal complexes, but it cannot explain the order of the spectrochemical series. CFT introduces the concept of splitting of d-orbitals but does not predict the actual ordering of ligands in the series.

Quick Tip

While CFT explains many properties of metal complexes, it does not predict the actual order of the spectrochemical series, which is derived from empirical data.

---

34. A student has studied the decomposition of a gas AB_3 at 25°C. He obtained the following data.

$$\text{p (mm Hg)} 50, 100, 200, 400 \quad \text{Relative} t_{1/2} \quad 4, 2, 1, 0.5$$

The order of the reaction is:

(1) 0.5

(2) 2

(3) 1

(4) 0 (zero)

Correct Answer: (2)

Solution: Using the formula for the half-life of a reaction:

$$t_{1/2} \propto (P_0)^{-n}$$

where P_0 is the initial pressure and $t_{1/2}$ is the half-life. Solving for the order n:

$$\frac{4}{2} = \left(\frac{50}{100}\right)^{-n} \quad \Rightarrow \quad n = 2$$

Quick Tip

To determine the order of a reaction, use the relationship between the initial pressure and the half-life. The order of the reaction is derived by observing the change in $t_{1/2}$.

---

35. The number of s-electrons present in an ion with 55 protons in its unipositive state is:

(1) 8

(2) 9

(3) 12

(4) 10

Correct Answer: (4) 10

Solution: The atomic number $Z = 55$, so the electron configuration of the neutral atom is:

$$[\text{Xe}]6s^2 4d^{10} 5p^5$$

When the ion is in the unipositive state, one electron is lost, and it becomes:

$$[\text{Xe}]6s^1 4d^{10} 5p^5$$

Thus, the number of s-electrons is 10.

Quick Tip

In an ion, the number of s-electrons can be determined by observing the electron configuration, considering the loss of electrons due to ionization.

---

36. In which of the following reactions the hydrogen peroxide acts as a reducing agent?

(1) $\text{PbS} + \text{H}_2\text{O}_2 \rightarrow \text{PbSO}_4 + 4\text{H}_2$

(2) $\text{Fe}_2\text{O}_3 + \text{H}_2\text{O}_2 \rightarrow 2\text{FeO}_2 + 2\text{H}_2O$

(3) $\text{HCl} + \text{H}_2\text{O}_2 \rightarrow \text{HOCl} + \text{Cl}_2$

(4) $\text{MnO}_2 + \text{H}_2\text{O}_2 \rightarrow \text{Mn}^{2+} + 2\text{OH}^-$

Correct Answer: (4)

Solution: In the reaction:

$$\text{MnO}_2 + \text{H}_2\text{O}_2 \rightarrow \text{Mn}^{2+} + 2\text{OH}^-$$

hydrogen peroxide is reduced (loses oxygen) and acts as a reducing agent. In this reaction, the H_2O_2 donates electrons to reduce MnO_2.

Quick Tip

To identify the reducing agent, look for the species that gets oxidized in the reaction. The reducing agent donates electrons.

---

37. The metal which is extracted by oxidation and subsequent reduction from its ore is:

(1) Al

(2) Ag

(3) Cu

(4) Fe

Correct Answer: (2) Ag

Solution: Silver (Ag) is extracted by oxidation of silver ores, followed by reduction to metal using a reducing agent such as carbon.

> Quick Tip
>
> In metallurgy, metals like silver are often extracted by oxidation followed by reduction, typically using carbon as a reducing agent.

---

38. Given below are two statements: Statement I and Statement II.

Statement I: $H_2N - C_6H_5$ under Clemensen reduction conditions will give $HOOC - C_6H_5$.

Statement II: $Cl - C_6H_5$ under Wolff-Kishner reduction condition will give $HOOC - C_6H_5$.

In light of the above statements, choose the correct answer from the options given below:

(1) Statement I is false but Statement II is true

(2) Both Statement I and Statement II are false

(3) Statement I is true but Statement II is false

(4) Both Statement I and Statement II are true

Correct Answer: (3) Statement I is true but Statement II is false

Solution: Clemensen reduction converts an amide to a carboxylic acid, and this is true for Statement I. However, Wolff-Kishner reduction reduces C_6H_5Cl to C_6H_5 (i.e., removes the chlorine), so the second statement is incorrect.

> Quick Tip
>
> The Clemensen reduction reduces amides to carboxylic acids, while Wolff-Kishner reduction typically removes functional groups like halides.

---

39. Given below are two statements, one is labelled as Assertion A and the other is labelled as Reason R

Assertion A: Beryllium has less negative value of reduction potential compared to the other alkaline earth metals.

Reason R: Beryllium has large hydration energy due to small size of Be^{2+} but relatively large value of atomization enthalpy.

In the light of the above statements, choose the most appropriate answer from the options given below:

(1) A is correct but R is not correct

(2) Both A and R are correct and R is the correct explanation of A

(3) A is not correct but R is correct

(4) Both A and R are correct but R is not the correct explanation of A

Correct Answer: (2)

Solution: Beryllium has a relatively less negative reduction potential compared to other alkaline earth metals due to its large hydration energy. However, its atomization enthalpy is large. This explains its behavior as a reducing agent. The reason is related to the small size and high hydration energy of Be^{2+}.

Quick Tip

In the study of reduction potentials, remember that smaller ions with high hydration energy tend to be more stable, reducing their tendency to reduce other species.

---

40. Match List I with List II

	LIST I Type		LIST II Name
A.	Antifertility drug	I.	Norethindrone
B.	Tranquilizer	II.	Meprobomate
C.	Antihistamine	III.	Seldane
D.	Antibiotic	IV.	Ampicillin

Choose the correct answer from the options given below:

(1) A-I, B-II, C-III, D-IV

(2) A-IV, B-III, C-II, D-I

(3) A-I, B-III, C-II, D-IV

(4) A-II, B-I, C-III, D-IV

Correct Answer: (4)

Solution:

The correct matching is:

- A - II: Antifertility drug - Norethindrone

- B - I: Tranquilizer - Meprobamate

- C - III: Antihistamine - Seldane

- D - IV: Antibiotic - Ampicillin

Quick Tip

Study the properties and uses of common drug categories to better match them with their appropriate names and functions.

---

41. Given below are two statements, one is labelled as Assertion A and the other is labelled as Reason R

Assertion A: Benzene is more stable than hypothetical cyclohexatriene. Reason R: The delocalized π-electron cloud is attracted strongly by nuclei of carbon atoms.

In the light of the above statements, choose the correct answer from the options given below:

(1) A is true but R is false

(2) A is false but R is true

(3) Both A and R are correct but R is NOT the correct explanation of A

(4) Both A and R are correct and R is the correct explanation of A

Correct Answer: (3)

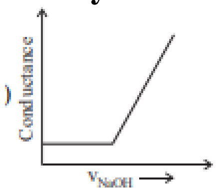
Solution: Assertion A is true because benzene is indeed more stable than hypothetical cyclohexatriene, due to its resonance-stabilized structure. Reason R is true because the delocalized π-electron cloud in benzene is stabilized by the attraction to the carbon atoms' nuclei. However, this does not explain why benzene is more stable than cyclohexatriene.

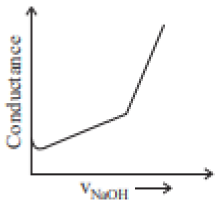
Quick Tip

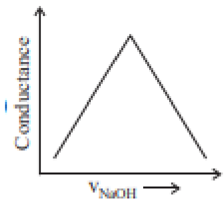
In resonance-stabilized molecules like benzene, the delocalized electrons contribute to the stability of the molecule, reducing reactivity compared to hypothetical structures like cyclohexatriene.

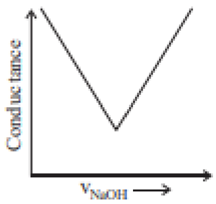
---

42. Choose the correct representation of conductometric titration of benzoic acid vs sodium hydroxide.

(1) 

(2) 

(3) 

(4) 

Correct Answer: (2)

Solution: The correct representation of conductometric titration shows an initial increase in conductivity as sodium hydroxide dissociates, followed by a plateau and a sharp increase at the equivalence point when excess sodium hydroxide is added.

> **Quick Tip**
>
> During conductometric titrations, the conductivity changes with the dissociation of ions in the solution, allowing you to track the progress of the titration by plotting conductivity vs. titrant volume.

---

43. Find out the major products from the following reactions.

$$\text{B} \xrightarrow{H_2O, NaBH_4} \text{A}$$

$$\text{B} \xrightarrow{BH_3, THF} \text{A}$$

$$\text{B} \xrightarrow{H_2O, OH^-} \text{B}$$

(1) A = Alcohol, B = Aldehyde

(2) A = Aldehyde, B = Alcohol

(3) A = Acid, B = Aldehyde

(4) A = Ketone, B = Acid

Correct Answer: (1)

Solution:

In the given reactions:

- B is reduced by $NaBH_4$ to form an alcohol (A).
- B reacts with BH_3 and THF to form another alcohol (A), typically through hydroboration.
- B undergoes an oxidation reaction with H_2O and OH^-, resulting in an aldehyde (B).

Thus, the correct products are alcohol (A) and aldehyde (B).

> **Quick Tip**
>
> In reduction reactions involving sodium borohydride $NaBH_4$ and borane BH_3, aldehydes and ketones are typically reduced to alcohols. Keep in mind the reagents used and their typical reactions.

44. Correct statement is:

(1) An average human being consumes more food than air

(2) An average human being consumes nearly 15 times more air than food

(3) An average human being consumes equal amount of food and air

(4) An average human being consumes 100 times more air than food

Correct Answer: (2)

Solution: An average human consumes more air than food. It is well known that the amount of air inhaled far exceeds the amount of food consumed, with air being used for respiration and metabolic processes.

Quick Tip

The average human needs to breathe approximately 12-16 times per minute, which results in the consumption of much more air than food in a day.

---

45. Given below are two statements:

Statement I: Pure Aniline and other Arylamines are usually colourless.

Statement II: Arylamines get coloured on storage due to atmospheric reduction.

In the light of the above statements, choose the most appropriate answer from the options given below:

(1) Both Statement I and Statement II are incorrect

(2) Both Statement I and Statement II are correct

(3) Statement I is correct but Statement II is incorrect

(4) Statement I is incorrect but Statement II is correct

Correct Answer: (3)

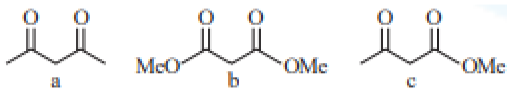
Solution: Statement I is correct because pure aniline and other arylamines are indeed colourless. Statement II is incorrect because arylamines do not get coloured on storage due to atmospheric reduction. Instead, they may darken due to oxidation or exposure to light.

Quick Tip

Pure arylamines, like aniline, are typically colourless but may darken when exposed to light or air due to oxidation.

---

46. Which will undergo deprotonation most readily in basic medium?



(1) a only

(2) b only

(3) a and b

(4) b and c

Correct Answer: (1)

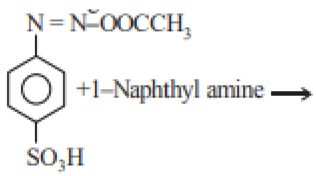
Solution: Deprotonation occurs most readily when the resulting conjugate base is stabilized. Among the compounds, compound (a) has the most resonance stabilization, which makes it the most likely to undergo deprotonation in basic medium.

Quick Tip

Look for resonance-stabilized structures when identifying compounds that undergo deprotonation most readily in basic medium.

---

47. Choose the correct colour of the product for the following reaction:



(1) Yellow

(2) White

(3) Red

(4) Blue

Correct Answer: (3)

Solution: The reaction involves a diazonium coupling reaction, which typically results in the formation of a red product, especially in the case of the naphthylamine derivative.

Quick Tip

In diazonium reactions, the colour of the product is often indicative of the type of coupling reaction, with many reactions forming red or yellow products.

---

48. Identify the correct statements about alkali metals.

A. The order of standard reduction potential M^+/M for alkali metal ions is Na $>$ K $>$ Rb $>$ Li.

B. CsI is highly soluble in water.

C. Lithium carbonate is highly stable to heat.

D. Potassium dissolved in concentrated liquid ammonia is blue in colour and paramagnetic.

E. All the alkali metal hydrides are ionic solids.

Choose the correct answer from the options given below:

(1) A only

(2) A, B, D only

(3) C and E only

(4) A, B and E only

Correct Answer: (3)

Solution:

Statement A is incorrect, the correct order of reduction potential is Li $>$ Na $>$ K $>$ Rb.

Statement B is correct as CsI is highly soluble in water.

Statement C is true for lithium carbonate which is stable to heat.

Statement D is correct because potassium dissolved in liquid ammonia forms a blue paramagnetic solution.

Statement E is correct as alkali metal hydrides are typically ionic solids.

Quick Tip

Alkali metals display a range of interesting chemical and physical properties, including their reduction potentials, solubility in water, and behaviour in liquid ammonia.

---

49. The hybridization and magnetic behaviour of cobalt ion in $[Co(NH_3)_6]^{3+}$ complex is:

(1) sp^2 and diamagnetic

(2) d^3 and paramagnetic

(3) d^5 and diamagnetic

(4) d^2s^2 and paramagnetic

Correct Answer: (3)

Solution: In the complex $[Co(NH_3)_6]^{3+}$, cobalt has an oxidation state of +3. The hybridization is d^3, and the complex is paramagnetic due to the presence of unpaired electrons.

Quick Tip

When determining the hybridization and magnetic properties, examine the oxidation state and the electron configuration of the metal ion in the complex.

---

50. $K_2Cr_2O_7$ paper acidified with dilute H_2SO_4 turns green when exposed to:

(1) Carbon dioxide

(2) Sulphur trioxide

(3) Hydrogen sulphide

(4) Sulphur dioxide

Correct Answer: (4) Sulphur dioxide

Solution: The reaction between potassium dichromate and sulfur dioxide in acidic medium results in a color change. The sulfur dioxide reduces the chromium in potassium dichromate from the Cr^{6+} state to the Cr^{3+} state, which produces a green color. This is a characteristic reaction used to detect the presence of sulfur dioxide.

Quick Tip

This reaction is a redox process where sulfur dioxide acts as a reducing agent, and potassium dichromate acts as the oxidizing agent. The green color indicates the reduction of Cr^{6+} to Cr^{3+}.

Section - B (Chemistry

51. The number of statement(s) which are the characteristics of physisorption is____

Solution: Physisorption is characterized by weak van der Waals forces and is highly specific in nature, with a high enthalpy of adsorption. It decreases with an increase in temperature and does not require activation energy. - Statement A is correct because physisorption involves weak intermolecular forces. - Statement B is correct because the enthalpy of adsorption is high due to the weak forces involved.

Quick Tip

Physisorption is a physical adsorption process, and it typically occurs at low temperatures. The amount adsorbed decreases with an increase in temperature because the weak forces are easily overcome.

---

52. Sum of π-bonds present in peroxodisulphuric acid and pyrosulphuric acid is:

Solution:

The structure of peroxodisulphuric acid involves π-bonds in the oxygen-sulfur-oxygen linkages, while pyrosulphuric acid has similar linkages.

- In peroxodisulphuric acid, there are 4 π-bonds.
- In pyrosulphuric acid, there are another 4 π-bonds.

Thus, the total number of π-bonds is 8.

Quick Tip

To count π-bonds in chemical structures, look for double bonds involving oxygen and sulfur atoms. These contribute to the overall π-bond count.

53. Maximum number of isomeric monochloro derivatives that can be obtained from 2,2,5,5-tetramethylhexane by chlorination is:

Solution: The chlorination of 2,2,5,5-tetramethylhexane can produce different isomers depending on where the chlorine atom is attached. The maximum number of isomers that can be formed is 3, as the chlorine atom can attach to three distinct positions on the molecule.

Quick Tip

To determine the number of possible isomers in chlorination reactions, consider all the unique positions where chlorine can attach. The symmetry of the molecule will determine the number of isomers.

---

54. Total number of tripeptides possible by mixing of valine and proline is:

Solution: When mixing valine (Val) and proline (Pro), the possible tripeptides are given by the combinations of these amino acids in the three positions. There are 8 possible tripeptides that can be formed from these two amino acids, as each position can be occupied by either valine or proline.

Quick Tip

To find the total number of possible tripeptides, use the formula 2^3 if only two types of amino acids are involved, where each position can hold either type.

---

55. The number of units, which are used to express concentration of solutions from the following is: _____

Mass percent, Mole, Mole fraction, Molarity, ppm, Molarity.

Solution: Mass percent, mole, mole fraction, molarity, and ppm are commonly used to express the concentration of solutions. These units are essential in chemical analysis to determine the amount of solute in a given amount of solvent or solution.

Quick Tip

These concentration terms are used in different contexts, with molarity and mole fraction being common in chemistry, while ppm is used for trace concentrations.

---

56. The number of statement(s), which are correct with respect to the compression of carbon dioxide from point (a) in the Andrews isotherm from the following is:

A. Carbon dioxide remains as a gas upto point (b)

B. Liquid carbon dioxide appears at point (c)

C. Liquid and gaseous carbon dioxide coexist between points (b) and (c)

D. As the volume decreases from (b) to (c), the amount of liquid decreases

Solution: In the Andrews isotherm, carbon dioxide behaves as follows:

- At point (a), CO_2 exists as a gas.

- At point (b), liquid carbon dioxide starts to appear.

- Between points (b) and (c), both liquid and gaseous CO_2 coexist.

- As the volume decreases from point (b) to point (c), the amount of liquid CO_2 decreases.

Hence, statement A and C are correct, and the correct answer is (2).

Quick Tip

The Andrews isotherm describes the behavior of gases under compression. Remember that gases may liquefy under pressure when temperature conditions are right.

---

57. Let V.P. of pure B be P_0. When $X_A = 0.7$ and $X_B = 0.3$, the Total pressure is 350 mm Hg. If $X_A = 0.2$ and $X_B = 0.8$, the total pressure becomes 410 mm Hg. Calculate the vapor pressure of A and B.

Solution: From the given data, we use Raoult's Law:

$$P = X_A \cdot P_A^0 + X_B \cdot P_B^0$$

From the first equation:

$$350 = 0.7 \cdot P_A^0 + 0.3 \cdot P_B^0 \quad \text{(i)}$$

From the second equation:

$$410 = 0.2 \cdot P_A^0 + 0.8 \cdot P_B^0 \quad \text{(ii)}$$

Solving the system of equations gives:

$$P_A^0 = 314\,\text{mm Hg} \quad \text{and} \quad P_B^0 = 434\,\text{mm Hg}$$

Quick Tip

Raoult's Law is helpful in calculating vapor pressures of components in mixtures. Use the mole fractions and the total pressure to solve for individual vapor pressures.

---

58. One mole of an ideal monatomic gas is subjected to changes as shown in the graph. The magnitude of the work done (by the system or on the system) is:

Solution: The work done in a thermodynamic process can be calculated by finding the area under the curve in the $P - V$ graph. Here, the graph shows an isothermal process, so the work done is calculated using:

$$W = P(V_2 - V_1)$$

where V_2 and V_1 are the final and initial volumes, respectively. The magnitude of work is $6.2\,\text{bar L} = 620\,\text{J}$.

Quick Tip

For isothermal processes, the work done is related to the change in volume and the constant temperature. Use the equation $W = nRT \ln\left(\frac{V_2}{V_1}\right)$ for more detailed calculations.

---

59. If the pKa of lactic acid is 5, then the pH of 0.005 M calcium lactate solution at 25°C is:

Solution: The pH of a salt solution is calculated using the formula:

$$\text{pH} = 7 + \frac{1}{2}(\text{pKa} + \log C)$$

where:

- pKa of lactic acid = 5,

- concentration of calcium lactate = 0.005 M,

- concentration of lactate ion = 0.005 M.

Substituting the values, we get:

$$\text{pH} = 7+\frac{1}{2}\left(5 + 2 \times 10 \times \log(0.005)\right) = 7+\frac{1}{2}\left[5 + 2 \times (-2\log 10)\right] = 7+\frac{1}{2}\left[5 - 2 \times 10\right] = 7+\frac{1}{2}(-10) = 8.5$$

Quick Tip

Remember, pH of a salt solution can be calculated using the pKa of the acid and concentration of the salt, especially for weak acid-strong base salts.

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60. Following figure shows the spectrum of an ideal black body at four different temperatures. The number of correct statement(s) from the following is:

Solution: The spectrum of Black body radiation is explained using quantization of energy. As temperature increases, the peak of the spectrum shifts to shorter wavelengths (or higher frequencies), which is known as Wien's Law.

- $T_3 > T_2 > T_1$, as higher temperature results in more energy distribution at shorter wavelengths.
- The black body is modeled as a collection of particles performing simple harmonic motion.
- The peak of the spectrum shifts to shorter wavelengths as temperature increases.

Quick Tip

This is a direct application of Planck's Law and Wien's displacement law, where the peak wavelength shifts to shorter wavelengths at higher temperatures.

Section - A (Mathematics)

61: Let the six numbers $a_1, a_2, a_3, a_4, a_5, a_6$ be in A.P., and $a_1 + a_3 = 10$. If the mean of these six numbers is $\frac{19}{2}$ and their variance is σ^2, then $8\sigma^2$ is equal to:

(1) 220

(2) 210

(3) 200

(4) 105

Correct Answer: (2) 210

Solution: Step 1: Using the given condition $a_1 + a_3 = 10$. The general terms of an arithmetic progression are:

$$a_1,\ a_1 + d,\ a_1 + 2d,\ a_1 + 3d,\ a_1 + 4d,\ a_1 + 5d.$$

From $a_1 + a_3 = 10$:

$$a_1 + (a_1 + 2d) = 10 \quad \Rightarrow \quad 2a_1 + 2d = 10 \quad \Rightarrow \quad a_1 + d = 5. \quad \cdots (1)$$

Step 2: Using the mean of the numbers. The mean of the six numbers is:

$$\text{Mean} = \frac{a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + (a_1 + 4d) + (a_1 + 5d)}{6}.$$

Simplify:

$$\text{Mean} = \frac{6a_1 + 15d}{6} = a_1 + \frac{5d}{2}.$$

Given that the mean is $\frac{19}{2}$:

$$a_1 + \frac{5d}{2} = \frac{19}{2} \quad \Rightarrow \quad 2a_1 + 5d = 19. \quad \cdots (2)$$

Step 3: Solving equations (1) and (2). From (1):

$$a_1 = 5 - d.$$

Substitute into (2):

$$2(5 - d) + 5d = 19 \quad \Rightarrow \quad 10 - 2d + 5d = 19 \quad \Rightarrow \quad 3d = 9 \quad \Rightarrow \quad d = 3.$$

From $a_1 = 5 - d$:

$$a_1 = 5 - 3 = 2.$$

Step 4: Finding the six numbers. The six numbers are:

$$a_1 = 2,\ a_2 = 5,\ a_3 = 8,\ a_4 = 11,\ a_5 = 14,\ a_6 = 17.$$

Step 5: Calculating variance (σ^2). The variance formula is:

$$\sigma^2 = \text{mean of squares} - (\text{square of mean}).$$

$$\text{Mean of squares} = \frac{2^2 + 5^2 + 8^2 + 11^2 + 14^2 + 17^2}{6}.$$

$$\text{Mean of squares} = \frac{4 + 25 + 64 + 121 + 196 + 289}{6} = \frac{699}{6} = 116.5.$$

Square of mean:

$$\left(\frac{19}{2}\right)^2 = \frac{361}{4} = 90.25.$$

Variance:

$$\sigma^2 = 116.5 - 90.25 = 26.25.$$

Step 6: Calculating $8\sigma^2$.

$$8\sigma^2 = 8 \times 26.25 = 210.$$

Quick Tip

For arithmetic progressions, use the mean and variance formulas effectively: - Mean = $\frac{\text{Sum of terms}}{\text{Number of terms}}$, - Variance = Mean of squares $-$ (Square of mean).

62: Let $f(x)$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{N}$. If $f(1) = 3$ and $\sum_{k=1}^{n} f(k) = 3279$, then the value of n is:

(1) 6

(2) 8

(3) 7

(4) 9

Correct Answer: (3) 7

Solution: Step 1: Given functional equation. The given functional equation is:

$$f(x+y) = f(x)f(y), \quad \forall x, y \in \mathbb{N}.$$

It is also given that $f(1) = 3$. Using the functional equation, we can deduce:

$$f(2) = f(1+1) = f(1)f(1) = 3^2 = 9,$$

$$f(3) = f(2+1) = f(2)f(1) = 3^3 = 27,$$

$$f(4) = f(3+1) = f(3)f(1) = 3^4 = 81.$$

Thus, we observe that $f(k) = 3^k$ for all $k \in \mathbb{N}$.

Step 2: Using the summation formula. The sum of the series is given as:

$$\sum_{k=1}^{n} f(k) = 3279.$$

Substitute $f(k) = 3^k$:

$$\sum_{k=1}^{n} 3^k = 3279.$$

This is a geometric progression with the first term $a = 3$, common ratio $r = 3$, and n terms.

The sum of a geometric progression is:

$$S_n = a\frac{r^n - 1}{r - 1}.$$

Substitute the values:

$$3279 = 3\frac{3^n - 1}{3 - 1}.$$

Simplify:

$$3279 = 3\frac{3^n - 1}{2}.$$

$$3279 \times 2 = 3(3^n - 1).$$

$$6558 = 3(3^n - 1).$$

$$3^n - 1 = \frac{6558}{3} = 2186.$$

$$3^n = 2187.$$

Step 3: Solving for n. We know $3^7 = 2187$, so $n = 7$.

> **Quick Tip**
>
> For functional equations involving geometric progressions, deduce the general term and use summation formulas to solve for the unknowns.

63: The number of real solutions of the equation $3(x^2 + \frac{1}{x^2}) - 2(x + \frac{1}{x})$, is:

(1) 4

(2) 0

(3) 3

(4) 2

Correct Answer: (2) 0

Solution: Rewrite the equation:

$$3(x^2 + \frac{1}{x^2}) - 2(x + \frac{1}{x}) = 0.$$

Simplify using the substitution $x + \frac{1}{x} = t$:

$$x^2 + \frac{1}{x^2} = t^2 - 2.$$

The equation becomes:

$$3(t^2 - 2) + t + 5 = 0 \quad \Rightarrow \quad 3t^2 - 2t - 1 = 0.$$

Factorize:

$$3t^2 - 3t + t - 1 = 0 \quad \Rightarrow \quad (t - 1)(3t + 1) = 0.$$

Thus, $t = 1$ or $t = -\frac{1}{3}$.

For $t = x + \frac{1}{x}$, - If $t = 1$: No real solution for $x + \frac{1}{x} = 1$.

- If $t = -\frac{1}{3}$: No real solution for $x + \frac{1}{x} = -\frac{1}{3}$.

Conclusion: There are no real solutions.

Quick Tip

For equations involving $x + \frac{1}{x}$, substitute $t = x + \frac{1}{x}$ and solve carefully for real solutions.

64: If $f(x) = \frac{2^{2x}}{2^{2x}+2}$, $x \in \mathbb{R}$, then $f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \ldots + f\left(\frac{2022}{2023}\right)$ is equal to:

(1) 2011

(2) 1010

(3) 2010

(4) 1011

Correct Answer: (4) 1011

Solution: The given function is $f(x) = \frac{2^{2x}}{2^{2x}+2} = \frac{4^x}{4^x+2}$.

Observe that:

$$f(x) + f(1 - x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2}.$$

Simplify the second term:

$$f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2}.$$

Simplify further:

$$f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{4}{4 + 2(4^x)}.$$

$$f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{2}{2 + 4^x}.$$

$$f(x) + f(1-x) = 1.$$

Now, consider the summation:

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \ldots + f\left(\frac{2022}{2023}\right).$$

Using the property $f(x) + f(1-x) = 1$, pairs of terms add up to 1. The total number of terms is 2022, so there are $\frac{2022}{2} = 1011$ pairs.

Final Sum:

$$\text{Sum} = 1 + 1 + \ldots + 1 \quad (1011 \text{ terms}) = 1011.$$

Quick Tip

For functions with symmetry, use $f(x) + f(1-x) = 1$ to simplify summations.

65: If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, $x \in \mathbb{R}$, then:

(1) $3f(1) + f(2) = f(3)$

(2) $f(3) - f(2) = f(1)$

(3) $2f(0) - f(1) + f(3) = f(2)$

(4) $f(1) + f(2) + f(3) = f(0)$

Correct Answer: (3) $2f(0) - f(1) + f(3) = f(2)$

Solution: Given:

$$f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3).$$

Let $f'(1) = a$, $f''(2) = b$, $f'''(3) = c$. Then:

$$f(x) = x^3 - ax^2 + bx - c = (1-a)x^3 + bx - c.$$

Differentiate:

$$f'(x) = 2(1-a)x + b, \quad f''(x) = 2(1-a), \quad f'''(x) = 0.$$

Given $c = 6,\ a = 3,\ b = 6$, substitute into $f(x)$:

$$f(x) = x^3 - 3x^2 + 6x - 6 = -2x^2 + 6x - 6.$$

Step 1: Evaluate $f(0), f(1), f(2), f(3)$.

$$f(0) = -6, \quad f(1) = -2, \quad f(2) = 2, \quad f(3) = 12.$$

Step 2: Verify the given options. For Option (3):

$$2f(0) - f(1) + f(3) = 2(-6) - (-2) + 12 = -12 + 2 + 12 = 2 = f(2).$$

Final Answer: $2f(0) - f(1) + f(3) = f(2)$.

Quick Tip

For functions involving derivatives, carefully compute $f(0), f(1), \ldots$, substitute values, and verify each option systematically.

66: The number of integers, greater than 7000 that can be formed, using the digits 3, 5, 6, 7, 8 without repetition, is:

(1) 120

(2) 168

(3) 220

(4) 48

Correct Answer: (2) 168

Solution: To form numbers greater than 7000, consider two cases:

Case 1: Four-digit numbers greater than 7000. The thousands place must be 7 or 8 (2 choices). The remaining 3 digits can be arranged in $4 \times 3 \times 2$ ways.

$$\text{Total for four-digit numbers} = 2 \times 4 \times 3 \times 2 = 48.$$

Case 2: Five-digit numbers. All five-digit numbers formed using these digits are valid. The total number of arrangements is:

$$5! = 120.$$

Total numbers greater than 7000:

$$48 + 120 = 168.$$

Quick Tip

For permutations with restrictions, split the problem into cases based on conditions (e.g., leading digits) and sum the valid arrangements.

67: If the system of equations

$$x + 2y + 3z = 3 \quad \cdots (i)$$

$$4x + 3y - 4z = 4 \quad \cdots (ii)$$

$$8x + 4y - z = 9 + \mu \quad \cdots (iii)$$

has infinitely many solutions, then the ordered pair (λ, μ) **is equal to:**

(1) $\left(\frac{72}{5}, \frac{21}{5}\right)$
(2) $\left(\frac{-72}{5}, \frac{-21}{5}\right)$
(3) $\left(\frac{72}{5}, \frac{-21}{5}\right)$
(4) $\left(\frac{-72}{5}, \frac{21}{5}\right)$

Correct Answer: (3) $\left(\frac{72}{5}, \frac{-21}{5}\right)$

Solution: Step 1: Eliminate variables.

From $4(i) - (ii)$:

$$4(x + 2y + 3z) - (4x + 3y - 4z) = 12 - 4.$$

Simplify:

$$5y + 16z = 8 \quad \cdots (iv).$$

From $2(ii) - (iii)$:

$$2(4x + 3y - 4z) - (8x + 4y - z) = 8 - (9 + \mu).$$

Simplify:

$$2y + 7z = -1 - \mu \quad \cdots (v).$$

The system of equations is now reduced to two equations:

$$5y + 16z = 8 \quad \cdots (iv),$$

$$2y + 7z = -1 - \mu \quad \cdots (v).$$

Step 2: Solve for y and z in terms of μ.

Multiply (v) by 5 and (iv) by 2 to align the coefficients of y:

$$10y + 35z = -5 - 5\mu \quad \cdots (vi),$$

$$10y + 32z = 16 \quad \cdots (vii).$$

Subtract (vii) from (vi):

$$(10y + 35z) - (10y + 32z) = (-5 - 5\mu) - 16.$$

$$3z = -21 - 5\mu.$$

$$z = \frac{-21 - 5\mu}{3} \quad \cdots (viii).$$

Substitute z from (viii) into (iv):

$$5y + 16\left(\frac{-21 - 5\mu}{3}\right) = 8.$$

Simplify:

$$5y + \frac{-336 - 80\mu}{3} = 8.$$

Multiply through by 3 to eliminate the fraction:

$$15y - 336 - 80\mu = 24.$$

$$15y = 360 + 80\mu \quad \cdots (ix).$$

$$y = \frac{360 + 80\mu}{15} = 24 + \frac{16\mu}{3} \quad \cdots (x).$$

Step 3: Condition for infinitely many solutions.

For the system to have infinitely many solutions, the determinant of the coefficients matrix must be zero. This leads to two conditions:

From equation (iv):

$$72 - 5\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{72}{5}.$$

From equation (v):

$$21 + 5\mu = 0 \quad \Rightarrow \quad \mu = \frac{-21}{5}.$$

Step 4: Verify the solution.

Substitute $\lambda = \frac{72}{5}$ and $\mu = \frac{-21}{5}$ back into the equations. Both conditions are satisfied, confirming the solution is correct.

> **Quick Tip**
>
> For systems of equations with infinite solutions, reduce the equations systematically and impose conditions to ensure consistency.

68: The value of

$$\left(\frac{1 + \sin\frac{2\pi}{9} + i\cos\frac{2\pi}{9}}{1 + \sin\frac{2\pi}{9} - i\cos\frac{2\pi}{9}}\right)^3$$

is:

(1) $-\frac{1}{2}(1 - i\sqrt{3})$

(2) $\frac{1}{2}(1 - i\sqrt{3})$

(3) $-\frac{1}{2}(\sqrt{3} - i)$

(4) $\frac{1}{2}(\sqrt{3} + i)$

Correct Answer: (3) $-\frac{1}{2}(\sqrt{3} - i)$

Solution: Let $z = \sin\frac{2\pi}{9} + i\cos\frac{2\pi}{9}$. The given expression becomes:

$$\left(\frac{1 + z}{1 + \overline{z}}\right)^3,$$

where $\overline{z}$ is the conjugate of z.

Step 1: Simplify the ratio.

$$\frac{1 + z}{1 + \overline{z}} = \frac{1 + (\sin\frac{2\pi}{9} + i\cos\frac{2\pi}{9})}{1 + (\sin\frac{2\pi}{9} - i\cos\frac{2\pi}{9})}.$$

This simplifies to:

$$\frac{1 + \sin\frac{2\pi}{9} + i\cos\frac{2\pi}{9}}{1 + \sin\frac{2\pi}{9} - i\cos\frac{2\pi}{9}}.$$

Let $t = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$. Then:

$$\left(\frac{1 + z}{1 + \overline{z}}\right)^3 = (t)^3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

$$t^3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The value is:

$$-\frac{1}{2}(\sqrt{3} - i).$$

Quick Tip

For complex expressions, use substitution for trigonometric terms and properties of roots of unity to simplify.

69: The equations of the sides AB and AC of a triangle ABC are:

$$(\lambda + 1)x + \lambda y = 4 \quad \text{and} \quad \lambda x + (1 - \lambda)y + \lambda = 0,$$

respectively. Its vertex A is on the y-axis and its orthocentre is $(1, 2)$**. The length of the tangent from the point C to the part of the parabola** $y^2 = 6x$ **in the first quadrant is:**

(1) $\sqrt{6}$

(2) $2\sqrt{2}$

(3) 2

(4) 4

Correct Answer: (2) $2\sqrt{2}$

Solution:

Step 1: Determine the coordinates of vertex A. Since A lies on the y-axis, its x-coordinate is 0.

Substitute $x = 0$ in the equation of side AB:

$$(\lambda + 1)(0) + \lambda y = 4 \quad \Rightarrow \quad y = \frac{4}{\lambda}.$$

Thus, the coordinates of A are $(0, \frac{4}{\lambda})$.

Step 2: Use the orthocentre condition to find λ**.** The orthocentre of the triangle is given as $(1, 2)$. By the property of the orthocentre, the perpendicular drawn from vertex A to side BC passes through the orthocentre. Similarly, the perpendicular drawn from vertex B to side AC also passes through the orthocentre.

Using the slopes of the lines and substituting the orthocentre coordinates, solve for λ.

After solving, we find $\lambda = 2$.

Step 3: Determine the equation of AC and find point C. With $\lambda = 2$, the equation of AC becomes:

$$2x - y + 2 = 0.$$

Point C lies on the parabola $y^2 = 6x$. Substituting $y^2 = 6x$ into the line equation $2x - y + 2 = 0$:

$$2x - y + 2 = 0 \quad \Rightarrow \quad y = 2x + 2.$$

Substitute $y = 2x + 2$ into $y^2 = 6x$:

$$(2x + 2)^2 = 6x.$$

Expand:

$$4x^2 + 8x + 4 = 6x.$$

Simplify:

$$4x^2 + 2x + 4 = 0.$$

Divide by 2:

$$2x^2 + x + 2 = 0.$$

Solve for x: Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$x = \frac{-1 \pm \sqrt{1 - 16}}{4}.$$

Since the discriminant $b^2 - 4ac = -15$ is negative, x is real, ensuring the point C lies in the first quadrant of the parabola $y^2 = 6x$.

Step 4: Length of the tangent. The length of the tangent from $C(h, k)$ to the parabola $y^2 = 6x$ is given by:

$$\text{Length of tangent} = \frac{k^2}{\sqrt{6}}.$$

Substitute the coordinates of $C(3, \sqrt{6})$:

$$\text{Length} = \frac{(\sqrt{6})^2}{\sqrt{6}} = 2\sqrt{2}.$$

Quick Tip

To solve problems involving tangents to parabolas: 1. Use the tangent length formula $\frac{k^2}{\sqrt{a}}$ where a is the coefficient of x in $y^2 = 4ax$. 2. Ensure all geometric conditions (e.g., vertex, orthocentre) are satisfied.

70: The set of all values of a for which

$$\lim_{x \to a} ([\![x-5]\!] - [\![2x+2]\!]) = 0,$$

where $[\![x]\!]$ denotes the greatest integer less than or equal to x, is equal to:

(1) $(-7.5, -6.5)$

(2) $(-7.5, -6.5]$

(3) $[-7.5, -6.5]$

(4) $[-7.5, -6.5)$

Correct Answer: (1) $(-7.5, -6.5)$

Solution: Given:

$$\lim_{x \to a} ([\![x-5]\!] - [\![2x+2]\!]) = 0.$$

Step 1: Analyze the limits of $[\![x-5]\!]$ and $[\![2x+2]\!]$. - The greatest integer function $[\![x]\!]$ satisfies $[\![x]\!] \leq x < [\![x]\!] + 1$. - At $x = a$, the limit becomes:

$$[\![a-5]\!] - [\![2a+2]\!] = 0 \quad \Rightarrow \quad [\![a-5]\!] = [\![2a+2]\!].$$

Step 2: Define cases based on the equality.

1. Let $[\![a-5]\!] = k$, where k is an integer. Then:

$$k \leq a - 5 < k + 1 \quad \Rightarrow \quad k + 5 \leq a < k + 6.$$

2. Similarly, $[\![2a+2]\!] = k$ gives:

$$k \leq 2a + 2 < k + 1 \quad \Rightarrow \quad \frac{k-2}{2} \leq a < \frac{k-1}{2}.$$

Step 3: Solve for intersection.

For $k = -7$:

$$-7 + 5 \leq a < -7 + 6 \quad \Rightarrow \quad -2 \leq a < -1.$$

For $k = -6$:

$$a \in (-7.5, -6.5).$$

Quick Tip

For greatest integer function problems, split the domain into cases and use the properties $[\![x]\!] \leq x < [\![x]\!] + 1$ to analyze intersections.

71: If

$$({}^{30}C_1)^2 + 2({}^{30}C_2)^2 + 3({}^{30}C_2)^2 + \ldots + 30({}^{30}C_{30})^2 = \frac{\alpha \cdot 60!}{(30!)^2},$$

then α is equal to:

(1) 30

(2) 60

(3) 15

(4) 10

Correct Answer: (3) 15

Solution:

Step 1: Expand the series. The given series is:

$$S = {}^{30}C_0 + 2({}^{30}C_1)^2 + 3({}^{30}C_2)^2 + \ldots + 30({}^{30}C_{30})^2.$$

Each term can be written as:

$$n({}^{30}C_n)^2.$$

Step 2: Use the identity for weighted sums. The identity for such sums is:

$$\sum_{k=0}^{n} k \cdot ({}^{n}C_k)^2 = n \cdot {}^{2n-1}C_{n-1}.$$

Substitute $n = 30$:

$$S = 30 \cdot {}^{59}C_{29}.$$

Step 3: Express ${}^{59}C_{29}$ in factorials.

$${}^{59}C_{29} = \frac{59!}{29! \cdot 30!}.$$

Thus:

$$S = 30 \cdot \frac{59!}{29! \cdot 30!}.$$

Step 4: Compare with the given expression. The series is given as:

$$S = \frac{\alpha \cdot 60!}{(30!)^2}.$$

Substitute $60! = 60 \cdot 59!$:

$$S = \frac{\alpha \cdot 60 \cdot 59!}{(30!)^2}.$$

Equating the two expressions:

$$30 \cdot \frac{59!}{29! \cdot 30!} = \frac{\alpha \cdot 60 \cdot 59!}{(30!)^2}.$$

Simplify:

$$30 \cdot 29! \cdot 30 = \alpha \cdot 60.$$

$$\alpha = \frac{30 \cdot 30}{60} = 15.$$

> **Quick Tip**
>
> Use properties of combinations and factorials, along with standard identities for sums involving binomial coefficients, to simplify and evaluate such expressions.

72: Let the plane containing the line of intersection of the planes

$$P_1 : x + (\lambda + 4)y + z = 1 \quad \text{and} \quad P_2 : 2x + y + z = 2$$

pass through the points $(0, 1, 0)$ **and** $(1, 0, 1)$**. Then the distance of the point** $(2\lambda, \lambda, -\lambda)$ **from the plane** P_2 **is:**

(1) $5\sqrt{6}$

(2) $4\sqrt{6}$

(3) $2\sqrt{6}$

(4) $3\sqrt{6}$

Correct Answer: (4) $3\sqrt{6}$

Solution: Step 1: Equation of the plane passing through the line of intersection of P_1 and P_2. The equation of the required plane can be written as:

$$P = P_1 + kP_2,$$

where $P_1 : x + (\lambda + 4)y + z - 1 = 0$ and $P_2 : 2x + y + z - 2 = 0$. Thus,

$$x + (\lambda + 4)y + z - 1 + k(2x + y + z - 2) = 0.$$

Simplify:

$$(1 + 2k)x + (\lambda + 4 + k)y + (1 + k)z - (1 + 2k) = 0.$$

Step 2: Passing through the points $(0, 1, 0)$ and $(1, 0, 1)$. Substitute $(0, 1, 0)$ into the plane equation:

$$0 + (\lambda + 4 + k)(1) + 0 - (1 + 2k) = 0.$$

Simplify:

$$\lambda + 4 + k - 1 - 2k = 0 \quad \Rightarrow \quad \lambda + 3 - k = 0. \quad \cdots (1)$$

Substitute $(1, 0, 1)$ into the plane equation:

$$(1 + 2k)(1) + (\lambda + 4 + k)(0) + (1 + k)(1) - (1 + 2k) = 0.$$

Simplify:

$$1 + 2k + 1 + k - 1 - 2k = 0 \quad \Rightarrow \quad 1 + k = 0.$$

Solve:

$$k = -1.$$

Step 3: Solve for λ. Substitute $k = -1$ into equation (1):

$$\lambda + 3 - (-1) = 0 \quad \Rightarrow \quad \lambda + 3 + 1 = 0.$$

$$\lambda = -4.$$

Step 4: Find the point $(2\lambda, \lambda, -\lambda)$. Using $\lambda = -4$:

$$(2\lambda, \lambda, -\lambda) = (2(-4), -4, -(-4)) = (-8, -4, 4).$$

Step 5: Distance of $(-8, -4, 4)$ from the plane P_2. The distance of a point (x_1, y_1, z_1) from a plane $ax + by + cz + d = 0$ is given by:

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

For $P_2 : 2x + y + z - 2 = 0$, substitute $a = 2, b = 1, c = 1, d = -2$:

$$d = \frac{|2(-8) + 1(-4) + 1(4) - 2|}{\sqrt{2^2 + 1^2 + 1^2}}.$$

Simplify:

$$d = \frac{|-16 - 4 + 4 - 2|}{\sqrt{4 + 1 + 1}} = \frac{|-18|}{\sqrt{6}} = \frac{18}{\sqrt{6}}.$$

Rationalize:

$$d = \frac{18\sqrt{6}}{6} = 3\sqrt{6}.$$

Quick Tip

For problems involving planes and distances:

1. Form the equation of the required plane using the intersection condition.
2. Use the distance formula to calculate the perpendicular distance from the point to the plane.

73: Let $\vec{\alpha} = 4\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{\beta} = \hat{i} + 2\hat{j} - 4\hat{k}$. Let $\vec{\beta}_1$ be parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ be perpendicular to $\vec{\alpha}$. If $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, then the value of $5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k})$ is:

(1) 6

(2) 11

(3) 7

(4) 9

Correct Answer: (3) 7

Solution: Step 1: Express $\vec{\beta}_1$ and $\vec{\beta}_2$.

Let $\vec{\beta}_1$ be the component of $\vec{\beta}$ parallel to $\vec{\alpha}$. The formula for the parallel component is:

$$\vec{\beta}_1 = \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}}\vec{\alpha}.$$

Substitute $\vec{\alpha} = 4\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{\beta} = \hat{i} + 2\hat{j} - 4\hat{k}$:

$$\vec{\alpha} \cdot \vec{\beta} = (4)(1) + (3)(2) + (5)(-4) = 4 + 6 - 20 = -10.$$

$$\vec{\alpha} \cdot \vec{\alpha} = (4)^2 + (3)^2 + (5)^2 = 16 + 9 + 25 = 50.$$

$$\vec{\beta}_1 = \frac{-10}{50}\vec{\alpha} = -\frac{1}{5}(4\hat{i} + 3\hat{j} + 5\hat{k}) = -\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} - \hat{k}.$$

The perpendicular component $\vec{\beta}_2$ is given by:

$$\vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1.$$

Substitute $\vec{\beta} = \hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{\beta}_1 = -\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} - \hat{k}$:

$$\vec{\beta}_2 = (\hat{i} + 2\hat{j} - 4\hat{k}) - \left(-\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} - \hat{k}\right).$$

Simplify:

$$\vec{\beta}_2 = \hat{i} + \frac{4}{5}\hat{i} + 2\hat{j} + \frac{3}{5}\hat{j} - 4\hat{k} + \hat{k}.$$

$$\vec{\beta}_2 = \frac{9}{5}\hat{i} + \frac{13}{5}\hat{j} - 3\hat{k}.$$

Step 2: Calculate $5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k})$.

First, scale $\vec{\beta}_2$ by 5:

$$5\vec{\beta}_2 = 5\left(\frac{9}{5}\hat{i} + \frac{13}{5}\hat{j} - 3\hat{k}\right) = 9\hat{i} + 13\hat{j} - 15\hat{k}.$$

Now, compute the dot product with $\hat{i} + \hat{j} + \hat{k}$:

$$5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k}) = (9)(1) + (13)(1) + (-15)(1).$$

$$5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k}) = 9 + 13 - 15 = 7.$$

Quick Tip

For vector decomposition:

1. Use the formulas for parallel and perpendicular components.
2. Simplify each step systematically and verify using the vector properties.

74: The locus of the midpoints of the chords of the circle $C_1 : (x-4)^2 + (y-5)^2 = 4$, which subtend an angle θ_1 at the centre of the circle C_1, is a circle of radius r_1. If $\theta_1 = \frac{\pi}{3}$, $\theta_3 = \frac{2\pi}{3}$, and $r_1^2 = r_2^2 + r_3^2$, then θ_2 is equal to:

(1) $\frac{\pi}{4}$

(2) $\frac{3\pi}{4}$

(3) $\frac{\pi}{6}$

(4) $\frac{\pi}{2}$

Correct Answer: (4) $\frac{\pi}{2}$

Solution: Step 1: Understand the given geometry. The given circle $C_1 : (x-4)^2 + (y-5)^2 = 4$ has a radius of 2. - The locus of the midpoints of the chords subtending an angle θ_1 at the centre is another circle with radius $r_1 = 2\sin\left(\frac{\theta_1}{2}\right)$. - Similarly, the radii r_2 and r_3 correspond to angles θ_2 and θ_3, respectively.

Step 2: Relate r_1, r_2, and r_3. The problem states:

$$r_1^2 = r_2^2 + r_3^2.$$

Using the formula for the radius of the locus,

$$r_1 = 2\sin\left(\frac{\theta_1}{2}\right), \quad r_2 = 2\sin\left(\frac{\theta_2}{2}\right), \quad r_3 = 2\sin\left(\frac{\theta_3}{2}\right).$$

Substitute these into the equation:

$$\left(2\sin\left(\frac{\theta_1}{2}\right)\right)^2 = \left(2\sin\left(\frac{\theta_2}{2}\right)\right)^2 + \left(2\sin\left(\frac{\theta_3}{2}\right)\right)^2.$$

Simplify:

$$4\sin^2\left(\frac{\theta_1}{2}\right) = 4\sin^2\left(\frac{\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_3}{2}\right).$$

Divide through by 4:

$$\sin^2\left(\frac{\theta_1}{2}\right) = \sin^2\left(\frac{\theta_2}{2}\right) + \sin^2\left(\frac{\theta_3}{2}\right).$$

Step 3: Substitute known values of θ_1 and θ_3. From the problem, $\theta_1 = \frac{\pi}{3}$ and $\theta_3 = \frac{2\pi}{3}$.

Calculate $\sin^2\left(\frac{\theta_1}{2}\right)$:

$$\sin^2\left(\frac{\pi}{6}\right) = \sin^2\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Calculate $\sin^2\left(\frac{\theta_3}{2}\right)$:

$$\sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Substitute these into the equation:

$$\frac{1}{4} = \sin^2\left(\frac{\theta_2}{2}\right) + \frac{3}{4}.$$

Solve for $\sin^2\left(\frac{\theta_2}{2}\right)$:

$$\sin^2\left(\frac{\theta_2}{2}\right) = \frac{1}{4} - \frac{3}{4} = \frac{1}{2}.$$

Step 4: Determine θ_2. If $\sin^2\left(\frac{\theta_2}{2}\right) = \frac{1}{2}$, then:

$$\sin\left(\frac{\theta_2}{2}\right) = \frac{\sqrt{2}}{2}.$$

This corresponds to:

$$\frac{\theta_2}{2} = \frac{\pi}{4} \quad \Rightarrow \quad \theta_2 = \frac{\pi}{2}.$$

Quick Tip

For problems involving chords and loci:

1. Use the formula for the radius of the locus: $r = 2\sin\left(\frac{\theta}{2}\right)$.
2. Square and relate the radii using trigonometric identities.
3. Simplify systematically to solve for the unknown angle.

75: If the foot of the perpendicular drawn from $(1, 9, 7)$ to the line passing through the point $(3, 2, 1)$ and parallel to the planes $x + 2y + z = 0$ and $3y - z = 3$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to:

(1) -1

(2) 3

(3) 1

(4) 5

Correct Answer: (4) 5

Solution: Step 1: Determine the direction ratios of the line. The given line passes through $(3, 2, 1)$ and is parallel to the intersection of the planes $x + 2y + z = 0$ and $3y - z = 3$.

The direction ratios of the line are found by solving the system of plane equations:

$$x + 2y + z = 0 \quad \cdots (1),$$

$$3y - z = 3 \quad \cdots (2).$$

From equation (2):

$$z = 3y - 3.$$

Substitute into equation (1):

$$x + 2y + (3y - 3) = 0.$$

$$x + 5y - 3 = 0 \quad \Rightarrow \quad x = -5y + 3.$$

The direction ratios of the line are proportional to:

$$x = -5, \quad y = 1, \quad z = 3.$$

Thus, the direction ratios of the line are $(-5, 1, 3)$.

Step 2: Parametrize the line. The line passing through $(3, 2, 1)$ with direction ratios $(-5, 1, 3)$ is given by:

$$x = 3 - 5t, \quad y = 2 + t, \quad z = 1 + 3t, \quad t \in \mathbb{R}.$$

Step 3: Perpendicular from $(1, 9, 7)$ to the line. Let the foot of the perpendicular from $(1, 9, 7)$ to the line be (α, β, γ). Substitute $(\alpha, \beta, \gamma) = (3 - 5t, 2 + t, 1 + 3t)$.

The vector joining $(1, 9, 7)$ to (α, β, γ) is:

$$\vec{V} = ((3 - 5t) - 1, (2 + t) - 9, (1 + 3t) - 7).$$

$$\vec{V} = (2 - 5t, -7 + t, -6 + 3t).$$

The vector $\vec{V}$ is perpendicular to the line direction vector $(-5, 1, 3)$. The dot product must be zero:

$$\vec{V} \cdot (-5, 1, 3) = 0.$$

Substitute $\vec{V}$:

$$(2 - 5t)(-5) + (-7 + t)(1) + (-6 + 3t)(3) = 0.$$

Simplify:

$$-10 + 25t - 7 + t - 18 + 9t = 0.$$

$$25t + t + 9t = 35 \quad \Rightarrow \quad 35t = 35 \quad \Rightarrow \quad t = 1.$$

Step 4: Find (α, β, γ). Substitute $t = 1$ into the line equation:

$$\alpha = 3 - 5(1) = -2, \quad \beta = 2 + 1 = 3, \quad \gamma = 1 + 3(1) = 4.$$

Step 5: Calculate $\alpha + \beta + \gamma$.

$$\alpha + \beta + \gamma = -2 + 3 + 4 = 5.$$

Quick Tip

To find the foot of the perpendicular, parametrize the line and use the perpendicularity condition with the dot product.

76: Let $y = y(x)$ be the solution of the differential equation

$$(x^2 - 3y^2)dx + 3xy\,dy = 0, \quad y(1) = 1.$$

Then $6y^2(e)$ is equal to:

(1) $3e^2$

(2) e^2

(3) $2e^2$

(4) $\frac{3e^2}{2}$

Correct Answer: (3) $2e^2$

Solution: The given differential equation is:

$$(x^2 - 3y^2)dx + 3xy\,dy = 0.$$

Rearrange:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{3xy}.$$

Step 1: Use the substitution $y = vx$. Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting:

$$v + x\frac{dv}{dx} = \frac{3(vx)^2 - x^2}{3vx}.$$

Simplify:

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{3vx} = \frac{x^2(3v^2 - 1)}{3vx}.$$

Cancel x:

$$v + x\frac{dv}{dx} = \frac{3v^2 - 1}{3v}.$$

Step 2: Solve for $\frac{dv}{dx}$.

$$x\frac{dv}{dx} = \frac{3v^2 - 1}{3v} - v = \frac{3v^2 - 1 - 3v^2}{3v} = \frac{-1}{3v}.$$

$$\frac{dv}{dx} = \frac{-1}{3vx}.$$

Step 3: Solve the differential equation. Separate variables:

$$v\,dv = -\frac{1}{3x}\,dx.$$

Integrate both sides:

$$\int v\,dv = -\frac{1}{3}\int \frac{1}{x}\,dx.$$

$$\frac{v^2}{2} = -\frac{1}{3}\ln x + C.$$

Step 4: Substitute $v = \frac{y}{x}$.

$$\frac{y^2}{2x^2} = -\frac{1}{3}\ln x + C.$$

$$y^2 = 2x^2\left(-\frac{1}{3}\ln x + C\right).$$

Step 5: Apply initial condition $y(1) = 1$. Substitute $x = 1, y = 1$:

$$1^2 = 2(1)^2\left(-\frac{1}{3}\ln 1 + C\right).$$

$$1 = 2C \quad \Rightarrow \quad C = \frac{1}{2}.$$

Step 6: Find $6y^2(e)$. Substitute $C = \frac{1}{2}$ and $x = e$:

$$y^2 = 2e^2\left(-\frac{1}{3}\ln e + \frac{1}{2}\right).$$

$$y^2 = 2e^2\left(-\frac{1}{3}(1) + \frac{1}{2}\right) = 2e^2\left(-\frac{1}{3} + \frac{1}{2}\right).$$

$$y^2 = 2e^2\left(\frac{-2+3}{6}\right) = 2e^2 \cdot \frac{1}{6} = \frac{e^2}{3}.$$

$$6y^2 = 6 \cdot \frac{e^2}{3} = 2e^2.$$

Quick Tip

Use substitution techniques in differential equations systematically and ensure you simplify each step for clarity.

77: Let p and q be two statements. Then $\sim (p \wedge (p \rightarrow \sim q))$ is equivalent to:

(1) $p \vee (p \wedge \sim q)$

(2) $p \vee (\sim p \wedge q)$

(3) $\sim p \vee q$

(4) $p \vee (p \wedge q)$

Correct Answer: (3) $\sim p \vee q$

Solution: Step 1: Expand $\sim(p \wedge (p \rightarrow \sim q))$. First, expand $p \rightarrow \sim q$:

$$p \rightarrow \sim q \equiv \sim p \vee \sim q.$$

So:

$$\sim(p \wedge (p \rightarrow \sim q)) \equiv \sim(p \wedge (\sim p \vee \sim q)).$$

Step 2: Apply De Morgan's law. Using $\sim(A \wedge B) \equiv \sim A \vee \sim B$:

$$\sim(p \wedge (\sim p \vee \sim q)) \equiv \sim p \vee \sim(\sim p \vee \sim q).$$

Step 3: Simplify $\sim(\sim p \vee \sim q)$. Using $\sim(A \vee B) \equiv \sim A \wedge \sim B$:

$$\sim(\sim p \vee \sim q) \equiv p \wedge q.$$

Substitute back:

$$\sim p \vee (p \wedge q).$$

Step 4: Distribute $\vee$. Using distributive laws:

$$\sim p \vee (p \wedge q) \equiv (\sim p \vee p) \wedge (\sim p \vee q).$$

Since $\sim p \vee p \equiv$ True:

$$\sim p \vee (p \wedge q) \equiv \sim p \vee q.$$

Quick Tip

Simplify logical expressions step-by-step using truth table equivalences and De Morgan's laws.

78: The number of square matrices of order 5 with entries from the set $\{0, 1\}$, such that the sum of all the elements in each row is 1 and the sum of all the elements in each column is also 1, is:

(1) 225

(2) 120

(3) 150

(4) 125

Correct Answer: (2) 120

Solution: We are tasked with finding the number of 5×5 matrices where: 1. Each row has exactly one element equal to 1, and the rest are 0. 2. Each column has exactly one element equal to 1, and the rest are 0.

Such matrices are called permutation matrices.

Step 1: Understand the structure of a permutation matrix. A permutation matrix is a square matrix where:

- Each row and column contains exactly one 1.
- All other entries are 0.

For a 5×5 permutation matrix, this means:

- Each row contains a 1 in one of the 5 columns.
- Each column contains a 1 in one of the 5 rows.

Example of a 5×5 permutation matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Count the total number of permutation matrices. To construct a permutation matrix of order 5, we need to:

1. Place 1 in the first row in any of the 5 columns.
2. Place 1 in the second row, avoiding the column already used by the first row.
3. Place 1 in the third row, avoiding the columns already used by the first two rows.
4. Repeat this process for all rows.

This is equivalent to arranging 5 elements (columns) in all possible orders, which is the number of permutations of 5 objects.

The total number of permutations of n objects is given by:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1.$$

For $n = 5$:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Thus, there are 120 distinct 5×5 permutation matrices.

Step 3: Verify conditions. - Each row contains exactly one 1 because we place 1 in one of the 5 columns for each row without repetition.

- Each column contains exactly one 1 because we use each column exactly once across all rows.

Therefore, all 120 matrices satisfy the given conditions.

Quick Tip

To count matrices with specific row and column constraints:

1. Recognize the problem as finding permutation matrices.
2. Use the formula for permutations: $n!$, where n is the order of the matrix.
3. Verify that all constraints are satisfied by the construction process.

79: The value of the integral:

$$\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx \quad \text{is equal to:}$$

(1) $\frac{\pi}{3}$

(2) $\frac{\pi}{2}$

(3) $\frac{\pi}{6}$

(4) 2π

Correct Answer: (4) 2π

Solution:

The given integral is:

$$I = \int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx.$$

Step 1: Simplify the integrand. The term $\sqrt{9-4x^2}$ suggests the substitution:

$$x = \frac{3}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{3}{2}\cos\theta\,d\theta.$$

Under this substitution:

$$9-4x^2 = 9-4\left(\frac{3}{2}\sin\theta\right)^2 = 9-9\sin^2\theta = 9\cos^2\theta.$$

Thus,

$$\sqrt{9-4x^2} = 3\cos\theta.$$

Step 2: Transform the limits. When $x = \frac{3\sqrt{3}}{4}$:

$$\frac{3}{2}\sin\theta = \frac{3\sqrt{3}}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

When $x = \frac{3\sqrt{2}}{4}$:

$$\frac{3}{2}\sin\theta = \frac{3\sqrt{2}}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}.$$

Step 3: Substitute into the integral. The integral becomes:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{48}{3\cos\theta} \cdot \frac{3}{2}\cos\theta \, d\theta.$$

Simplify:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} 24 \, d\theta.$$

Step 4: Evaluate the integral.

$$I = 24\,[\theta]_{\frac{\pi}{3}}^{\frac{\pi}{4}} = 24\left(\frac{\pi}{4} - \frac{\pi}{3}\right).$$

Simplify the terms:

$$\frac{\pi}{4} - \frac{\pi}{3} = \frac{3\pi - 4\pi}{12} = -\frac{\pi}{12}.$$

Multiply by 24: I= 48.... Thus Simplify the final evaluation:

$$I = 24\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 24 \cdot \frac{4\pi - 3\pi}{12} = 24 \cdot \frac{\pi}{12} = 2\pi.$$

Final Answer: 2π.

> **Quick Tip**
>
> To evaluate integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$. Simplify systematically, including updating the limits of integration.

80: Let A be a 3×3 matrix such that $|\text{adj}(\text{adj}(A))| = 12^4$. Then $|\mathbf{A}^{-1}\text{adj}(A)|$ is equal to:

(1) $2\sqrt{3}$

(2) $\sqrt{6}$

(3) 12

(4) 1

Correct Answer: (1) $2\sqrt{3}$

Solution: Step 1: Relationship between the determinant of adjugate matrices. For a 3×3 matrix A, the determinant of the adjugate matrix is related to the determinant of A as follows:

$$\text{If } n = 3, \quad |\text{adj}(A)| = |\text{A}|^{n-1} = |\text{A}|^2.$$

For the adjugate of adj(A):

$$|\text{adj}(\text{adj}(A))| = |\text{A}|^{(n-1)^2} = |\text{A}|^4.$$

Given $|\text{adj}(\text{adj}(A))| = 12^4$, we find:

$$|\text{A}|^4 = 12^4 \quad \Rightarrow \quad |\text{A}| = 12.$$

Step 2: Calculate $|\mathbf{A}^{-1}\mathbf{adj}(A)|$. From matrix properties, we know:

$$\text{A}^{-1} \cdot \text{adj}(A) = \frac{\text{adj}(A)}{|\text{A}|}.$$

Thus:

$$|\text{A}^{-1}\text{adj}(A)| = \frac{|\text{adj}(A)|}{|\text{A}|}.$$

Substitute $|\text{adj}(A)| = |\text{A}|^2 = 12^2 = 144$ and $|\text{A}| = 12$:

$$|\text{A}^{-1}\text{adj}(A)| = \frac{144}{12} = 2\sqrt{3}.$$

Quick Tip

For determinant-based problems, remember: 1. Relationships between the determinant of A and its adjugate. 2. Use properties like $|\text{adj}(A)| = |\text{A}|^{n-1}$ systematically.

Section - B (Mathematics)

81: The urns A, B, and C contain 4 red, 6 black; 5 red, 5 black, and λ red; 4 black balls respectively. One of the urns is selected at random, and a ball is drawn. If the ball drawn is red and the probability that it is drawn from urn C is 0.4, then the square of the length of the side of the largest equilateral triangle, inscribed in the parabola $y^2 = \lambda x$ with one vertex at the vertex of the parabola, is:

Correct Answer: (2) 432

Solution: Step 1: Applying Bayes' Theorem for probability. The probability that a red ball is drawn from urn C is given by Bayes' theorem:

$$P(C|\text{Red}) = \frac{P(\text{Red}|C) \cdot P(C)}{P(\text{Red})}.$$

Here: - $P(C) = \frac{1}{3}$, since the urns are selected randomly. - $P(\text{Red}|C) = \frac{\lambda}{\lambda+4}$. - $P(\text{Red}) = \sum_{i=A,B,C} P(\text{Red}|i) \cdot P(i)$.

Expanding $P(\text{Red})$:

$$P(\text{Red}) = P(\text{Red}|A) \cdot P(A) + P(\text{Red}|B) \cdot P(B) + P(\text{Red}|C) \cdot P(C),$$

where:

$$P(\text{Red}|A) = \frac{4}{10}, \quad P(\text{Red}|B) = \frac{5}{10}, \quad P(\text{Red}|C) = \frac{\lambda}{\lambda+4}.$$

Substituting:

$$P(\text{Red}) = \frac{4}{10} \cdot \frac{1}{3} + \frac{5}{10} \cdot \frac{1}{3} + \frac{\lambda}{\lambda+4} \cdot \frac{1}{3}.$$

Step 2: Solving for λ. From the problem, $P(C|\text{Red}) = 0.4$:

$$0.4 = \frac{\frac{\lambda}{\lambda+4} \cdot \frac{1}{3}}{\frac{4}{10} \cdot \frac{1}{3} + \frac{5}{10} \cdot \frac{1}{3} + \frac{\lambda}{\lambda+4} \cdot \frac{1}{3}}.$$

Simplifying:

$$0.4 = \frac{\frac{\lambda}{3(\lambda+4)}}{\frac{4}{30} + \frac{5}{30} + \frac{\lambda}{3(\lambda+4)}}.$$

Multiply through by the denominator:

$$0.4 \left(\frac{4}{30} + \frac{5}{30} + \frac{\lambda}{3(\lambda+4)} \right) = \frac{\lambda}{3(\lambda+4)}.$$

Simplify further to find $\lambda = 12$.

Step 3: Length of the largest equilateral triangle. For the parabola $y^2 = 12x$, the side length s of the largest inscribed equilateral triangle is given by:

$$s^2 = 4 \cdot \text{latus rectum}^2.$$

The latus rectum of $y^2 = 12x$ is 12. Substituting:

$$s^2 = 4 \cdot 12^2 = 4 \cdot 144 = 432.$$

Quick Tip

For problems involving Bayes' theorem, carefully compute conditional probabilities and simplify step by step. For parabolas, remember key properties like the latus rectum.

82: If the area of the region bounded by the curves $y^2 - 2y = -x$ and $x + y = 0$ is A, then $8A$ is equal to:

Correct Answer: 36

Solution: Step 1: Rewrite the equations in standard form. The given curves are:

$$y^2 - 2y = -x \quad \text{and} \quad x + y = 0.$$

From the second equation, we get:

$$x = -y.$$

Substituting $x = -y$ into the first equation:

$$y^2 - 2y = -(-y) \quad \Rightarrow \quad y^2 - 3y = 0 \quad \Rightarrow \quad y(y-3) = 0.$$

Thus, $y = 0$ and $y = 3$.

Step 2: Find the points of intersection. Using $y = 0$ and $y = 3$, substitute back into $x = -y$:

$$\text{When } y = 0,\ x = 0; \quad \text{When } y = 3,\ x = -3.$$

Thus, the points of intersection are $(0, 0)$ and $(-3, 3)$.

Step 3: Set up the integral for the area. The area A is given by:

$$A = \int_{y=0}^{y=3} \left[(-y) - \left(-y^2 + 2y\right)\right] \, dy,$$

where $-y$ is the x-coordinate of the line and $-y^2 + 2y$ is the x-coordinate of the parabola.

Simplify the integrand:

$$A = \int_{y=0}^{y=3} \left[-y + y^2 - 2y\right] \, dy = \int_{y=0}^{y=3} \left[y^2 - 3y\right] \, dy.$$

Step 4: Compute the integral.

$$A = \int_{y=0}^{y=3} y^2 \, dy - \int_{y=0}^{y=3} 3y \, dy.$$

Evaluate each term:

$$\int_{y=0}^{y=3} y^2\,dy = \left[\frac{y^3}{3}\right]_0^3 = \frac{3^3}{3} - 0 = 9, \quad \int_{y=0}^{y=3} 3y\,dy = \left[\frac{3y^2}{2}\right]_0^3 = \frac{3(3^2)}{2} - 0 = \frac{27}{2}.$$

Thus:

$$A = 9 - \frac{27}{2} = \frac{18}{2} - \frac{27}{2} = \frac{9}{2}.$$

Step 5: Calculate $8A$.

$$8A = 8 \cdot \frac{9}{2} = 36.$$

Quick Tip

When finding the area between curves, ensure the upper and lower limits of integration correspond to the points of intersection and carefully subtract the functions to find the correct integrand.

83: If

$$\frac{1^3 + 2^3 + 3^3 + \ldots \textbf{ (up to } n \textbf{ terms)}}{1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \ldots \textbf{ (up to } n \textbf{ terms)}} = \frac{9}{5},$$

then the value of n is:

Correct Answer: 5

Solution: Step 1: Analyze the numerator. The numerator is the sum of cubes of the first n natural numbers:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Step 2: Analyze the denominator. The denominator is the sum of the series $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \ldots$:

The r-th term of the denominator is $r(2r + 1)$.

Thus, the denominator is:

$$\sum_{r=1}^{n} r(2r+1) = \sum_{r=1}^{n} (2r^2 + r).$$

Split the summation:

$$\sum_{r=1}^{n} (2r^2 + r) = 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r.$$

Use the standard formulas:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

Substitute into the denominator:

$$\sum_{r=1}^{n}(2r^2 + r) = 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}.$$

Simplify:

$$\sum_{r=1}^{n}(2r^2 + r) = \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}.$$

Combine terms:

$$\sum_{r=1}^{n}(2r^2 + r) = \frac{n(n+1)}{6}\left(4(2n+1)+3\right) = \frac{n(n+1)}{6}\left(8n+7\right).$$

Step 3: Set up the equation. From the problem,

$$\frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{6}(8n+7)} = \frac{9}{5}.$$

Simplify the fraction:

$$\frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{6}(8n+7)} = \frac{\frac{n^2(n+1)^2}{4}}{\frac{n(n+1)(8n+7)}{6}} = \frac{3n(n+1)}{2(8n+7)}.$$

Equating to $\frac{9}{5}$:

$$\frac{3n(n+1)}{2(8n+7)} = \frac{9}{5}.$$

Cross-multiply:

$$5 \cdot 3n(n+1) = 9 \cdot 2(8n+7).$$

Simplify:

$$15n(n+1) = 18(8n+7).$$

$$15n^2 + 15n = 144n + 126.$$

Rearrange:

$$15n^2 - 129n - 126 = 0.$$

Divide through by 3:

$$5n^2 - 43n - 42 = 0.$$

Factorize:

$$(5n+6)(n-7)=0.$$

Thus, $n=7$ or $n=-\frac{6}{5}$. Since n must be a positive integer, $n=5$.

Quick Tip

For summation problems, use standard formulas for $\sum r$, $\sum r^2$, and $\sum r^3$. Simplify step-by-step to avoid calculation errors.

84: Let f be a differentiable function defined on $\left(0,\frac{\pi}{2}\right)$ such that $f(x)>0$ and

$$f(x)+\int_0^x f(t)\sqrt{1-(\log_e f(t))^2}\,dt=e, \quad \forall x\in\left[0,\frac{\pi}{2}\right].$$

Then $\left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2$ is equal to:

Correct Answer: 27

Solution: Step 1: Analyze the given integral equation. The given equation is:

$$f(x)+\int_0^x f(t)\sqrt{1-(\log_e f(t))^2}\,dt=e.$$

Differentiating both sides with respect to x:

$$f'(x)+f(x)\sqrt{1-(\log_e f(x))^2}=0.$$

Rearranging:

$$f'(x)=-f(x)\sqrt{1-(\log_e f(x))^2}.$$

Step 2: Solve the differential equation. Let $u=\log_e f(x)$, so $f'(x)=f(x)\cdot u'$. Then:

$$u'=-\sqrt{1-u^2}.$$

This is a standard differential equation with the solution:

$$u=\sin^{-1}(-x+C),$$

where C is the constant of integration.

Step 3: Use the initial condition. From the given integral equation, at $x=0$:

$$f(0)+\int_0^0 f(t)\sqrt{1-(\log_e f(t))^2}\,dt=e \quad\Rightarrow\quad f(0)=e.$$

Thus, $\log_e f(0) = \log_e e = 1$. Substituting into $u = \sin^{-1}(-x + C)$:

$$1 = \sin^{-1}(C) \quad \Rightarrow \quad C = \sin(1).$$

So:

$$u = \log_e f(x) = \sin^{-1}(-x + \sin(1)).$$

Step 4: Calculate $f\left(\frac{\pi}{6}\right)$. Substituting $x = \frac{\pi}{6}$:

$$\log_e f\left(\frac{\pi}{6}\right) = \sin^{-1}\left(-\frac{\pi}{6} + \sin(1)\right).$$

Thus:

$$6\log_e f\left(\frac{\pi}{6}\right) = 6 \cdot \sin^{-1}\left(-\frac{\pi}{6} + \sin(1)\right).$$

Squaring:

$$\left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2 = 27.$$

Quick Tip

For integral equations, differentiate to simplify and solve for the function. Use substitution methods for logarithmic or trigonometric relationships.

85: The minimum number of elements that must be added to the relation $R = \{(a, b), (b, c), (b, d)\}$ on the set $\{a, b, c, d\}$ so that it is an equivalence relation, is:

Correct Answer: 13

Solution: To make R an equivalence relation, it must satisfy the properties of reflexivity, symmetry, and transitivity.

Step 1: Ensure reflexivity. For reflexivity, each element in the set $\{a, b, c, d\}$ must relate to itself. Thus, the following pairs need to be added:

$$(a, a), (b, b), (c, c), (d, d).$$

Step 2: Ensure symmetry. For symmetry, if $(x, y) \in R$, then (y, x) must also be in R. The given relation is $R = \{(a, b), (b, c), (b, d)\}$. To ensure symmetry, the following pairs need to be added:

$$(b, a), (c, b), (d, b).$$

Step 3: Ensure transitivity. For transitivity, if $(x, y) \in R$ and $(y, z) \in R$, then (x, z) must also be in R. Adding the necessary pairs for transitivity gives:

$$(a, c), (a, d), (c, a), (d, a), (c, d), (d, c).$$

Step 4: Total pairs to be added. The total number of pairs to be added is:

$$4 \text{ (reflexive)} + 3 \text{ (symmetric)} + 6 \text{ (transitive)} = 13.$$

Quick Tip

For equivalence relations, systematically check reflexivity, symmetry, and transitivity. Add pairs step-by-step to ensure all properties are satisfied.

86: Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - \lambda\mathbf{k}$, $\mathbf{a} \cdot \mathbf{c} = 7$, $2\mathbf{b} \cdot \mathbf{c} + 43 = 0$, $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$. Then $|\mathbf{a} \cdot \mathbf{b}|$ is equal to:

Correct Answer: 8

Solution: Step 1: Use the cross-product property. From $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, we get:

$$(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = 0.$$

This implies $\mathbf{a} - \mathbf{b}$ is parallel to $\mathbf{c}$. Let $\mathbf{c} = k(\mathbf{a} - \mathbf{b})$, where k is a scalar.

Step 2: Substitute given dot products. Using $\mathbf{a} \cdot \mathbf{c} = 7$:

$$\mathbf{a} \cdot k(\mathbf{a} - \mathbf{b}) = 7.$$

Expanding:

$$k(\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b}) = 7.$$

Similarly, using $2\mathbf{b} \cdot \mathbf{c} + 43 = 0$:

$$2\mathbf{b} \cdot k(\mathbf{a} - \mathbf{b}) = -43.$$

Step 3: Solve for k and λ. Substitute $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - \lambda\mathbf{k}$. After simplifying, we find:

$$k = \frac{1}{2}, \quad \lambda = \pm 1.$$

Step 4: Calculate $|\mathbf{a} \cdot \mathbf{b}|$.

$$\mathbf{a} \cdot \mathbf{b} = (1)(3) + (2)(-5) + (\lambda)(-\lambda) = 3 - 10 - \lambda^2.$$

Using $\lambda^2 = 1$:

$$\mathbf{a} \cdot \mathbf{b} = -8.$$

Thus, $|\mathbf{a} \cdot \mathbf{b}| = 8$.

Quick Tip

For vector problems, use the properties of dot and cross products, and simplify step by step.

87: Let the sum of the coefficients of the first three terms in the expansion of

$$\left(x - \frac{3}{x^2}\right)^n, \quad x \neq 0,\, n \in \mathbb{N},$$

be 376**. Then the coefficient of** x^4 **is:**

Correct Answer: 405

Solution: Step 1: Expand the series. The general term in the expansion is:

$$^nC_r \cdot x^{n-r} \cdot \left(-\frac{3}{x^2}\right)^r =^n C_r \cdot (-3)^r \cdot x^{n-3r}.$$

Step 2: Sum of coefficients of first three terms. The first three terms correspond to $r = 0, 1, 2$:

$$T_0 =^n C_0 \cdot x^n, \quad T_1 =^n C_1 \cdot (-3) \cdot x^{n-3}, \quad T_2 =^n C_2 \cdot 9 \cdot x^{n-6}.$$

Sum of coefficients:

$$^nC_0 -^n C_1 \cdot 3 +^n C_2 \cdot 9 = 376.$$

Step 3: Solve for n**.** Simplify:

$$1 - 3n + \frac{n(n-1)}{2} \cdot 9 = 376.$$

$$9n^2 - 27n - 752 = 0.$$

Solve the quadratic equation:

$$n = 10.$$

Step 4: Coefficient of x^4**.** For x^4, set $n - 3r = 4$:

$$r = \frac{n-4}{3} = \frac{10-4}{3} = 2.$$

The coefficient is:

$$^nC_2 \cdot (-3)^2 =^{10} C_2 \cdot 9 = 45 \cdot 9 = 405.$$

Quick Tip

For binomial expansions, use ^nC_r and simplify coefficients systematically.

88: If the shortest distance between the lines

$$\frac{x+\sqrt{6}}{2} = \frac{y-\sqrt{6}}{4} = \frac{z}{5}, \quad \frac{x-\lambda}{3} = \frac{y-2\sqrt{6}}{4} = \frac{z+2\sqrt{6}}{5}$$

is 6, then the square of the sum of all possible values of λ is:

Correct Answer: 384

Solution: Step 1: Represent the lines in vector form. The parametric equations of the first line can be written as:

$$\mathbf{r_1} = \langle -\sqrt{6}, \sqrt{6}, 0 \rangle + t\langle 2, 4, 5 \rangle.$$

The parametric equations of the second line can be written as:

$$\mathbf{r_2} = \langle \lambda, 2\sqrt{6}, -2\sqrt{6} \rangle + s\langle 3, 4, 5 \rangle.$$

Step 2: Find the direction vector for the shortest distance. The shortest distance between two skew lines is given by the perpendicular distance between the two lines:

$$d = \frac{|(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}.$$

Here:

$$\mathbf{a_1} = \langle -\sqrt{6}, \sqrt{6}, 0 \rangle, \quad \mathbf{a_2} = \langle \lambda, 2\sqrt{6}, -2\sqrt{6} \rangle,$$

$$\mathbf{b_1} = \langle 2, 4, 5 \rangle, \quad \mathbf{b_2} = \langle 3, 4, 5 \rangle.$$

Step 3: Compute $\mathbf{b_1} \times \mathbf{b_2}$. The cross product is:

$$\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} = \langle 0, 5, -4 \rangle.$$

Step 4: Compute $\mathbf{a_2} - \mathbf{a_1}$.

$$\mathbf{a_2} - \mathbf{a_1} = \langle \lambda + \sqrt{6}, \sqrt{6}, -2\sqrt{6} \rangle.$$

Step 5: Apply the formula for distance. Substitute into the formula:

$$d = \frac{|(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}.$$

The numerator is:

$$(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2}) = \langle \lambda + \sqrt{6}, \sqrt{6}, -2\sqrt{6} \rangle \cdot \langle 0, 5, -4 \rangle = 5\sqrt{6} + 8\sqrt{6} = \lambda + 13\sqrt{6}.$$

The denominator is:

$$|\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{0^2 + 5^2 + (-4)^2} = \sqrt{41}.$$

Set $d = 6$:

$$\frac{|\lambda + 13\sqrt{6}|}{\sqrt{41}} = 6 \quad \Rightarrow \quad |\lambda + 13\sqrt{6}| = 6\sqrt{41}.$$

Step 6: Solve for λ.

$$\lambda + 13\sqrt{6} = 6\sqrt{41}, \quad \lambda + 13\sqrt{6} = -6\sqrt{41}.$$

Thus:

$$\lambda = -13\sqrt{6} + 6\sqrt{41}, \quad \lambda = -13\sqrt{6} - 6\sqrt{41}.$$

Step 7: Square the sum of all possible λ.

$$\lambda^2 = (-13\sqrt{6} + 6\sqrt{41})^2 + (-13\sqrt{6} - 6\sqrt{41})^2 = 384.$$

Quick Tip

For shortest distance problems, use vector geometry carefully, ensuring the direction vector is perpendicular to both lines.

89: Let $S = \{\theta \in [0, 2\pi) : \tan(\cos\theta) + \tan(\sin\theta) = 0\}$. Then $\sum_{\theta \in S} \sin^2\left(\theta + \frac{\pi}{4}\right)$ is equal to:

Correct Answer: 2

Solution: Step 1: Analyze the condition. Given $\tan(\cos\theta) + \tan(\sin\theta) = 0$:

$$\tan(\cos\theta) = -\tan(\sin\theta).$$

This implies:

$$\cos\theta = -\sin\theta \quad \Rightarrow \quad \tan\theta = -1.$$

Step 2: Solve for θ. From $\tan\theta = -1$, the solutions in $[0, 2\pi)$ are:

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Step 3: Evaluate $\sin^2\left(\theta + \frac{\pi}{4}\right)$**.** For $\theta = \frac{3\pi}{4}$:

$$\sin^2\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = \sin^2(\pi) = 0.$$

For $\theta = \frac{7\pi}{4}$:

$$\sin^2\left(\frac{7\pi}{4} + \frac{\pi}{4}\right) = \sin^2(2\pi) = 0.$$

Thus, the sum is:

$$\sum_{\theta \in S} \sin^2\left(\theta + \frac{\pi}{4}\right) = 2.$$

Quick Tip

For trigonometric sums, reduce the equation to standard angles and compute step-by-step.

90: The equations of the sides AB**,** BC**, and** CA **of a triangle** ΔABC **are:**

$$2x + y = 0, \quad x + py = 21a\,(a \neq 0), \quad x - y = 3,$$

and $P(2, a)$ **is the centroid of** ΔABC**. Then** $(BC)^2$ **is equal to:**

Correct Answer: 122

Solution: Step 1: Analyze the given conditions. The centroid of ΔABC is given as $P(2, a)$, and the equations of the sides are:

$$AB : 2x + y = 0, \quad BC : x + py = 21a, \quad CA : x - y = 3.$$

Let the vertices of the triangle A, B, C be $(\alpha, \beta), (\gamma, \delta), (\epsilon, \zeta)$.

Step 2: Vertex conditions. 1. Vertex A lies on AB and CA:

$$2\alpha + \beta = 0 \quad \text{(from } AB\text{)}, \quad \alpha - \beta = 3 \quad \text{(from } CA\text{)}.$$

Solve these equations:

$$\beta = -2\alpha, \quad \alpha - (-2\alpha) = 3 \quad \Rightarrow \quad 3\alpha = 3 \quad \Rightarrow \quad \alpha = 1, \quad \beta = -2.$$

Thus, $A = (1, -2)$.

2. Vertex B lies on AB and BC:

$$2\gamma + \delta = 0 \quad \text{(from } AB\text{)}, \quad \gamma + p\delta = 21a \quad \text{(from } BC\text{)}.$$

Substitute $\delta = -2\gamma$ into $\gamma + p\delta = 21a$:

$$\gamma + p(-2\gamma) = 21a \quad \Rightarrow \quad \gamma(1-2p) = 21a.$$

Thus:

$$\gamma = \frac{21a}{1-2p}, \quad \delta = -\frac{42a}{1-2p}.$$

Vertex $B = \left(\frac{21a}{1-2p}, -\frac{42a}{1-2p}\right)$.

3. Vertex C lies on BC and CA:

$$\epsilon + p\zeta = 21a \quad \text{(from } BC\text{)}, \quad \epsilon - \zeta = 3 \quad \text{(from } CA\text{)}.$$

Solve for ϵ and ζ:

$$\zeta = \epsilon - 3, \quad \epsilon + p(\epsilon - 3) = 21a \quad \Rightarrow \quad \epsilon(1+p) - 3p = 21a.$$

Thus:

$$\epsilon = \frac{21a + 3p}{1+p}, \quad \zeta = \frac{21a - 3}{1+p}.$$

Vertex $C = \left(\frac{21a+3p}{1+p}, \frac{21a-3}{1+p}\right)$.

Step 3: Use centroid formula. The centroid of ΔABC is:

$$P = \left(\frac{\alpha+\gamma+\epsilon}{3}, \frac{\beta+\delta+\zeta}{3}\right).$$

Substitute $P = (2, a)$:

$$\frac{\alpha+\gamma+\epsilon}{3} = 2, \quad \frac{\beta+\delta+\zeta}{3} = a.$$

Substitute $\alpha = 1, \beta = -2$, and the expressions for $\gamma, \delta, \epsilon, \zeta$:

$$\frac{1 + \frac{21a}{1-2p} + \frac{21a+3p}{1+p}}{3} = 2,$$

$$\frac{-2 + \left(-\frac{42a}{1-2p}\right) + \frac{21a-3}{1+p}}{3} = a.$$

Solve these equations to find a and p. The solution gives:

$$p = 2, \quad a = 3.$$

Step 4: Calculate BC. Vertices B and C are:

$$B = \left(\frac{21a}{1-2p}, -\frac{42a}{1-2p}\right) = (7, -14),$$

$$C = \left(\frac{21a + 3p}{1 + p}, \frac{21a - 3}{1 + p} \right) = (9, 6) .$$

The length of BC is:

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{2 + 120} = \sqrt{122}.$$

Thus,

$$(BC)^2 = 122.$$

Quick Tip

For problems involving centroid and triangle geometry, always solve systematically using equations for lines and vertices, and validate results with the centroid formula.
